Chapter – 7
PHASORS ALGEBRA

Vectors, in general, may be located anywhere in space. We have restricted ourselves thus for to vectors which are all located in one plane (co planar vectors), but they may still be anywhere in that plane. Such general vectors are referred to as free vectors. Our primary interest in vectors, however, relates to their application in the solution of AC circuits. For this purpose, we do not require the generality of free vectors, and we may restrict ourselves still further to vectors which all originate from a fixed point in the plane (i.e., the origin of our coordinate axes) and whose direction rotates about this point. Such vectors are called phasors.

The value of dealing with phasors, rather than vector in general, lies in the fact that phasors can be represented by complex numbers, with j (imaginary number with $j = \sqrt{-1}$) interpreted as an operator. Thus all the special mathematics of vectors, in the case of phasors, becomes simply a matter of the arithmetic of complex numbers

7.2 J as an Operator:

An operator is a symbol for a mathematical .operation. We have defined the imaginary number j ($j = i = \sqrt{-1}$), and from it built up the system of imaginary and complex numbers. In this chapter, we shift our view point slightly and consider j as an operator which is going to operate on real numbers.

Let us look at graphical behaviour of a real number which operated upon by j repeatedly. In fig. 1, we show the axis of real numbers (R axis) horizontal and the axis of real numbers affected by j (j axis) vertical to it or at right angles to it. Starting with the real number 2, we multiply by (i.e., operate with) j once and arrive at 2j. Multiplying by j again, we arrive at $2j^2 = -2$, A third multiplication by j yields $2j^3 = -2j$. The fourth multiplication by j yields $2j^4 = 2$, which brings us back to our staring point.

From- this example, we note that-the graphical effect of j as an operator is to rotate a point counter clockwise (CCW) about the origin, along a circle of constant radius, through an of 90°.

Fig.1
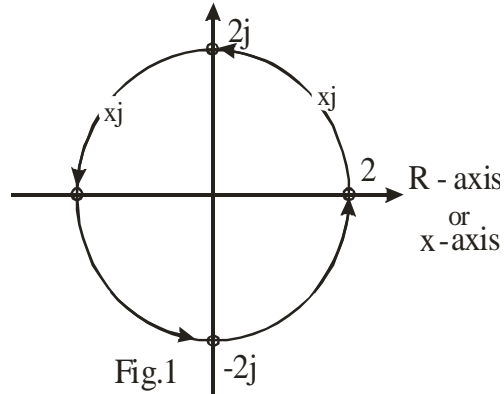
If we consider a quantity of the form a + jb (a and b real numbers) as representing a vector E whose tail is fixed at the origin and whose point is located by the coordinates (a, ib), the effect of operating on this vector E with j is to rotate it counter clockwise 90° from its initial position. Operating with j twice i.e., $j^2E = - E$, rotates the original vector E, through 180° counter clockwise. Operating with j three times., i.e, $j^3E = - jE$, rotates the original vector through 270^0 counter clockwise, which is equivalent to rotating through 90° clockwise. Four successive operations with j i.e, $j^4E=E$, rotates the original vector through 360° counter clockwise, which is the same vector E. These rotations are shown in Fig. 2.

Fig.2
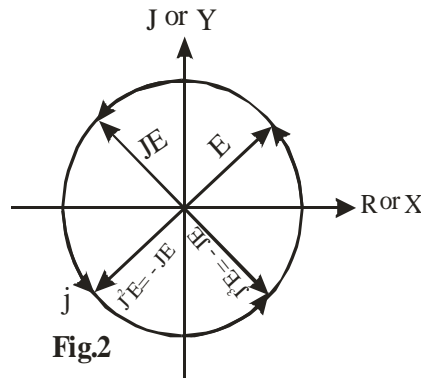
For vectors which rotates about some fixed point in a plane (called phasors), this concept of 'j' leads to a neat and simple algebraic way of performing vector operations.

7.3   Mathematical Representation of Phasors
(or complex numbers):

A Phasor can be represented graphically in the various forms such as:

(i)   Rectangular or Cartesian form

(ii)   Trigonometric and polar form.

(iii) Exponential form.

(i)   Rectangular or Cartesian Form:

A Phasor Z can be expressed in terms of its x-component 'x' and y-component 'y' as shown in Fig.3.

Mathematically it is written as,
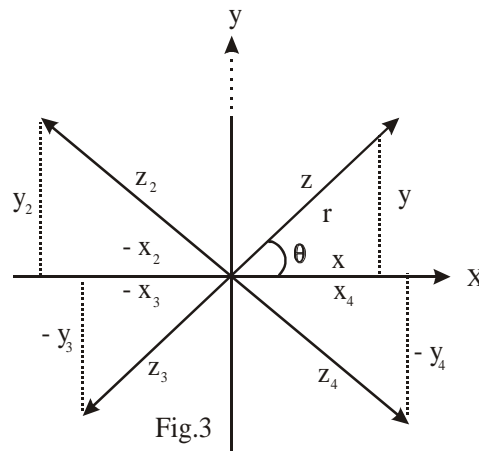
$$Z = x + j\,y$$

where $j = \sqrt{-1}$, known as operator, it indicates that the component y is perpendicular to the component x. In the Phasor or complex number

$Z = x + jy$, where x and y are called **real and imaginary** part of Z respectively. But in Electrical Engineering these are known as **in-phase**

(or active)and **quardrature**(or reactive) component respectively. The Phasor or complex numbers (or vectors) are shown in Fig. 3 and represented as,

$$z_2 = -x_2 + jy_2$$
$$z_3 = -x_3 - jy_2$$

and $\qquad z_4 = \quad x_4 - jy_2$

Fig.3

The numerical value (or magnitude) of Z is denoted by r or |Z| or |x + jy|, and is given by

$$r = |Z| = |x + jy| = \sqrt{x^2 + y^2}$$

The argument or amplitude of Z, denoted by arg (Z), is an angle θ with the positive x-axis, and is given by

$$\theta = \tan^{-1}\frac{y}{x}$$

Note: In mathematics $\sqrt{-1}$ is denoted by i, but in electrical engineering j is adopted because the letter 'j' is used for representing current.

(ii)   Trigonometric and Polar Form:

From Fig.4, we see that

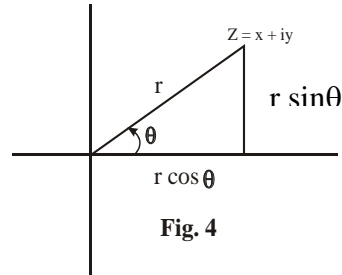$$\text{Cos } \theta = \frac{x}{r} \qquad \text{and} \qquad \text{Sin } \theta = \frac{y}{r}$$

Hence,   x = r Cos θ   and   y = r sin θ

Therefore,   the complex number

Z = x + jy

       = r Cos θ + j r sin θ

Z = r (cos θ + j sin θ)

Fig. 4

The general form of this equation is

Z = r (cos θ ± j Sin θ)

This is called the Trigonometric form of the complex number Z.

If we simply write r(cos θ + j sin θ) = r ∠ θ

Then   Z = r ∠ θ

In general,   Z = r ∠ ± θ

This is called the polar form (or Modulus argument form) of the complex number Z. Trigonometric and polar forms are the same, but the polar form is simply a short hand or symbolic style of writing the Trigonometric form.

(iii)   Exponential Form:

A very interesting and useful relation was discovered by the great Swiss mathematician Euler. Stated as an equation

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This equation is known as Euler's equation

(The derivation of this relationship is given at the end of this chapter).

If we apply this relationship to the trigonometric form of a complex number Z, then

$$Z = r(\cos\theta + j\sin\theta)$$
$$Z = r\,e^{j\theta}$$

In general, $Z = \pm\, r\,e^{j\theta}$

This relation is very useful for multiplication and division of complex numbers.

Hence, we get $Z = x + jy = r(\cos\theta + j\sin\theta) = r\angle\theta = r\,e^{j\theta}$

7.4 Conjugate Complex Numbers:

Two complex numbers are called the conjugate of each other if their real parts are equal and their imaginary parts differ only in sign. The conjugate of a complex number $Z = x + iy$, is denoted by $\bar{z}$ and is given as

$\bar{Z} = x - iy$.

Example 1:   Express the following in polar form:

(a)   (b) $4 - j5$ (c) $-1 - j$

Solution:

(a) $1 + j\sqrt{3}$

Here $x = 1,$ $y = \sqrt{3}$

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ$$

So, polar form is $=$ $r\angle\theta = 2\angle 60^\circ$

Its trigonometric form is $=$ $r(\cos\theta + j\sin\theta)$

$= 2(\cos 60^\circ + j\sin 60^\circ)$

(b) $4 - j5$

Here $x = 4,$ $y = -5$

$$r = \sqrt{(4)^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}$$

$$\theta = \tan^{-1}\left(-\frac{5}{4}\right) = 308°\ 40'$$

Hence, polar form is $r \angle \theta = \sqrt{41} \angle 308°\ 40'$

(c) $-1 - j$

Here $x = -1,$ $y = -1$

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-5}{-4}\right) = 225°$$

Hence, $r \angle \theta = \sqrt{2}$ $\angle 225°$

Example 2:   Express the following in rectangular form.

   (a) $10 \angle 3.5°$ **(b)** $450 \angle 94°$ **(c)** $12.3 \angle -45°$

Solution:

   (a) $10 \angle 3.5°$ $= 10(\cos 3.5° + j \sin 3.5°)$

                                        $= 10 (0.998 + j\ 0.06)$

                                        $= 9.98 + j\ 0.61$

   (b) $450 \angle 94°$ $= 450 (\cos 94° + j\ 94°)$

                                        $= 450(-0.698 + j\ 0.998)$

                                        $= 31.4 + j\ 449.1$

   (c) $12.3 \angle 45°$ $= 12.3 (\cos (-45°) + j (-45°))$

                                        $= 12.3(0.707 + j\ 0.707)$

                                        $= 8.696 - j\ 8.696$

Example 3:   Given that $z = 2e^{-j\pi}$, write the other forms.

Solution:

   Here $r = 2,$ $\theta = -\dfrac{\pi}{6} = -30°$

   So, polar form is $r \angle \theta = 2 \angle -30°$

   Trigonometric form is

            $r = (\cos \theta + j \sin \theta)$ $= 2 (\cos (-30°) + j \sin (-30°))$

                                        $= 2 (\cos 30° - j \sin 30°)$

Rectangular form is

$$x + jy = 2 (0.866 - j\, 0.5)$$
$$= 1.632 - j$$

7.5 Addition and Subtraction of Complex Numbers (Or vectors):

Addition and subtraction of complex numbers can be performed conveniently only when both numbers are in the rectangular form. Suppose we are given two complex numbers:

$$Z_1 = x_1 + jy_1 \quad , \quad Z_2 = x_2 + jy_2$$

(i) Addition: $Z = Z_1 + Z_2 \quad = (x_1 + jy_1) + (x_2 + jy_2)$
$$= (x_1 + x_2) + j\,(y_1 + y_2)$$

The magnitude of $Z = |Z| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$

The argument or amplitude of Z is $\theta = \tan^{-1}\left(\dfrac{y_1 + y_2}{x_1 + x_2}\right)$

(ii)Subtraction : $Z = Z_1 - Z_2 \quad = (x_1 + jy_1) - (x_2 + jy_2)$
$$= (x_1 - x_2) + j\,(y_1 - y_2)$$

The magnitude of $Z = |Z| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

The argument or amplitude of Z is $\theta = \tan^{-1}\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right)$

7.6 Multiplication and Division :

1- Multiplication:

(i)In Rectangular Form:

Since $Z = Z_1\, Z_2 \quad = (x_1 + jy_1)\,(x_2 + jy_2)$
$$= x_1 x_2 + y_1 y_2\, j^2 + j x_1\, y_2 + j x_2\, y_1$$
$$= (x_1 x_2 - y_1 y_2) + j\,(x_1\, y_2 + x_2\, y_1)$$

The magnitude of $Z = |Z| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + j(x_1 y_2 + x_2 y_1)^2}$

The argument or amplitude of $Z = \theta = \tan^{-1}\left(\dfrac{x_1 y_2 + x_2 y_1}{x_1 x_2 - y_1 y_2}\right)$

(ii)In Polar Form:

Since

$$Z_1 = x_1 + jy_1 \quad = r_1 \angle \theta_1 \quad = \quad r_1 e^{j\theta_1}$$

$$Z_2 = x_2 + iy_2 \quad = r_2 \angle \theta_2 \quad = \quad r_2 e^{j\theta_2}$$

$$Z_1 Z_2 = r_1 \angle \theta_1 \times r_2 \angle \theta = r_1 e^{j\theta_1} \times r_2 e^{j\theta_2}$$

$$= \quad r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$Z_1 Z_2 \quad = r_1 r_2 \angle \theta_1 + \theta_2$$

(iii)In Trigonometric Form:

Since, $Z_1 Z_2 \quad = r_1 r_2 \angle \theta_1 + \theta_2$

$$Z_1 Z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + j \sin((\theta_1 + \theta_2))]$$

2- Division

(i) In Rectangular Form:

Since $\quad \dfrac{Z_1}{Z_2} = \dfrac{x_1 + jy_1}{x_2 + jy_2}$

Multiplying and divide by the conjugate $z_2 - jy_2$ in order to made the denominator real.

$$\frac{Z_1}{Z_2} \quad = \frac{x_1 + jy_1}{x_2 + jy_2} \times \frac{x_2 - jy_2}{x_2 - jy_2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + j (x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Which is $x + jy$ form

Note :

Generally, the result be expressed in the form $x + jy$

(ii)In Polar Form:

Since $Z_1 = x_1 + jy_1 = r_1 \angle \theta_1 = r_1 e^{j\theta 1}$

$Z_2 = x_2 + iy_2 = r_2 \angle \theta_2 = r_2 e^{j\theta 2}$

So $\dfrac{Z_1}{Z_2} = \dfrac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \dfrac{r_1 e^{j\theta 1}}{r_2 e^{j\theta 2}}$

$= \dfrac{r_1}{r_2} e^{j(1\theta_1 - \theta_2)}$

$\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2} \angle \theta_1 - \theta_2$

(iii)In Trigonometric Form:

Since $\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2} \angle \theta_1 - \theta_2$

$\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2} [\text{Cos } (\theta_1 - \theta_2) + j \text{ Sin } (\theta_1 - \theta_2)]$

Example 4:   Find the product of $1 + j$, $1 + j\sqrt{3}$, $\sqrt{3} - j$

and express the result in polar form, trigonometry form and rectangular form.

Solution:

First we express each number is polar form.

$1 + j \qquad\qquad = \sqrt{2} \angle 45^o$

$1 + j\sqrt{3} \qquad\quad = 2 \angle 60^o$

and $\sqrt{3} - j \qquad\qquad = 2 \angle -30^o$

Hence, the product of the three numbers is

$\sqrt{2} \angle 45^o \times 2\angle 60^o \times 2 \angle -30^o$

$= \sqrt{2} \times 2 \times 2 \angle 45^o + 60^o - 30^o$

$= 4 \sqrt{2} \angle 75^o$-------------------------------- **polar form.**

Or $= 4 \sqrt{2} \ (\cos 76^o + j \sin 75^o)$-----**trigonometry form.**

Or $= 4 \sqrt{2} \ (0.259 + j\,0.966)$

$= 1.\,465 + j\,5.465$----------------- **rectangular form.**

Example 5: Given $Z_1 = 8\angle -30^o$ and $Z_2 = 2\angle -60^o$ find $\dfrac{Z_1}{Z_2}$ and

express the result in polar form, exponential form, trigonometry form and rectangular form.

Solution:

Since $\quad \dfrac{Z_1}{Z_2} \quad = \dfrac{8\angle -30^o}{2\angle -60^o}$

$$= \dfrac{8}{2} \angle -30^o + 60^o$$

$$= 4\angle 30^o \text{-- \textbf{polar form.}}$$

$Z \quad = \quad r\angle\theta = 4\,e^{\,j\,\pi/6}\text{----------------------------\textbf{exponential form.}}$

$Z \quad = 4\,(\cos 30^0 + j\sin 30^0)\text{------------------ \textbf{trigonometry form.}}$

$Z \quad = 4(0.866 + j0.5)$

$\quad\quad = 3.464 + j0.5 \text{-----------------------------\textbf{rectangular form.}}$

Example 6: Simplify $\dfrac{3+5j}{4+3j}$ to the form a + ib.

Solution: $\quad \dfrac{3+5i}{4+3j}$

Multiply and divide by conjugate $4 - 3j$ of denominator

so, $\qquad \dfrac{3+5j}{4+3j} \times \dfrac{4-3j}{4-3j}$

$$= \dfrac{(12+15)+j\,(20-9)}{16+9}$$

$$= \dfrac{27+j\,11}{25}$$

$$= \dfrac{27}{25} + j\dfrac{11}{25}$$

$$= 1.08 + j\,0.44$$

Example 7: **Perform the indicated operation and given the result in rectangular form ,polar form ,exponential form and trigonometry form.**

$$\dfrac{(1+\sqrt{3}\,j)\,(\sqrt{3}+j)}{1+j}$$

Solution:

$$Z = \frac{(1 + j\sqrt{3})(\sqrt{3} + j)}{1 + j} = \frac{(\sqrt{3} - \sqrt{3}) + 4i}{1 + j}$$

$$= \frac{4j}{1 + j} = \frac{4j}{1 + j} \times \frac{1 - j}{1 - j}$$

$$= \frac{4 + 4j}{1 + 1} = 2 + 2j \text{------------------ \textbf{rectangular form.}}$$

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = 45^\circ$$

Hence, $Z = r\angle\theta = 2\sqrt{2} \angle 45^\circ$ --------------------**polar form.**

$Z = r\,e^{j\theta} = 2\sqrt{2}\,e^{j\pi/4}$ ---------------**exponential form.**

$Z = 2\sqrt{2}(\cos 45^0 + j\sin 45^0)$---------- **trigonometry form.**

Example 8:   **Using $Z = \cos\theta + j\,\text{Sin}\,\theta$ and $Z^3 = \text{Cos}^3\theta + j\sin 3\theta$, by expanding $(\cos\theta + j\,\text{Sin}\,\theta)^3$ and equating real and imaginary parts show that:**

(a)   Cos $3\theta = 4\cos^3\theta - 3\cos\theta$

(b)   Sin $3\theta = 3\sin\theta - 4\sin^3\theta$

Solution:

Since $\cos 3\theta + j\sin 3\theta = Z^3$

So, $\cos 30 + j\,\text{Sin}\,30 = (\cos\theta + j\sin\theta)^3$

Expand by binomial theorem

$= \cos^3\theta + 3\cos^2\theta\,(j\sin\theta) + 3\cos\theta\,(j\sin\theta)^2 + (j\sin\theta)^3$

Put $j^2 = -1$ and $j^3 = -j$

$= \cos^3\theta + j\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + j(3\cos^2\theta\sin\theta - \sin^3\theta)$

Comparing real and imaginary parts on both sides, we get

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$

$= \cos^2\theta - 3\cos\theta(1 - \cos^2\theta)$

Cos $3\theta = 4\cos^3\theta - 3\cos\theta$

(b)   Sin 3θ =3 cos²θ sin θ - sin³ θ

$$=3(1 - \sin^2 θ) \sin θ - \sin^3 θ)$$

$$= 3 \sin θ - 4 \sin^3 θ$$

7.8 Powers and Roots of the Complex Numbers (Vectors): (De Moivre's Theorem)

If we square the complex number

$$Z = r (\cos θ + j \sin θ)$$

We get, $Z^2 = [r (\cos θ + j \sin θ)] [r(\cos θ + j \sin θ)]$

$$= r^2 (\cos 2 θ + j \sin 2 θ)$$

Further more

$$Z^3 = Z (Z^2) = [r (\cos θ + j \sin θ)] [r^2 (\cos 2θ + j \sin 2θ)]$$

$$= r^3 (\cos 3θ + j \sin 3θ)$$

A repeated application of this process leads to the following theorem if,

$$Z = r (\cos θ + j \sin θ) = r \angle θ$$

Then $Z^n = r^n (\cos n θ + j \sin n θ) = r^n \angle n θ$

And $Z^{1/n} = r^{1/n} (\cos \dfrac{θ}{n} + j \sin \dfrac{θ}{n})$

Or $\sqrt[n]{Z} = \sqrt[n]{r} (\cos \dfrac{θ}{n} + j \sin \dfrac{θ}{n}) = \sqrt[n]{r} \angle \dfrac{θ}{n}$

Which is the nth roots of Z.

Note:

The fact that

$$Z^{1/n} = r^{1/n} \angle \frac{θ}{n} = r^{1/n} \angle \frac{θ + K360^o}{n}$$

Where,   K = 0, ± 1, ± 2,…………enable use to find n distinct nth roots of the complex number by assigning to K the values 0, 1, 2, 3, ………n – 1.

Example 9:   Simplify   (a) $(\sqrt{3} + j)^7$

     (b) $\sqrt{3 - j4}$ and express the result in a + jb form

Solution:

(a) $\sqrt{3} + j = 2 \angle 30^\circ$,   which is a polar form

Then $(\sqrt{3} + j)^7 = (2\angle 30^\circ)^7 = 2^7 \angle 7\,(30^\circ)$

$= 128 \angle 210^\circ$

$= 128 \left[(\cos 210^\circ + j \sin 210^\circ\right]$

$= 128 \left(-\dfrac{\sqrt{3}}{2} - j\dfrac{1}{2}\right)$

$= 64\,(-\sqrt{3} - j)$

$= -64\sqrt{3} - j64$

(c)   Express first $3 - 4j$ in polar form:

$3 - j4 = 5 \angle -53.13^\circ$

So, $\sqrt{3 - j4} = (5 \angle -53.13^\circ)^{1/2}$

$= \sqrt{5} \angle \dfrac{-53.13^0 + k\,360^0}{2}$,   where $k = 0\,,\,1$

Hence the roots are

$= \sqrt{5} \angle \dfrac{-53.13^0}{2}$ and $\sqrt{5} \angle \dfrac{-53.13^0 + 360^0}{2}$

$\sqrt{5} \angle -26.57^\circ$ $= \sqrt{5} \angle 153.03^0$

$= \sqrt{5}\,(\cos(-26.57^\circ) + j \sin(-26.57^\circ)$ $= \sqrt{5}\,(\cos 153.03^0 + j\sin 153.03^0)$

$= \sqrt{5}\,(0.89 - j\,0.45)$ $= \sqrt{5}\,(-0.89 + j\,0.45)$

$= 2 - j$ $= -2 + j$

Example 10: Find the four fourth roots of $Z = 2 + \sqrt{3}\,j$ in the form $r \angle \theta$.

Solution:

Write $Z \quad = 2 + j2\sqrt{3}$ in polar form.

$Z \quad = 2 + 2\sqrt{3}\,j = 4\angle 60^\circ$

Now,   the fourth roots of Z is

$\omega = Z^{¼} = (4 \angle 60^\circ)^{¼} \qquad = 4^{¼} \angle \dfrac{60^\circ}{4}$

$$\omega_k = 4^{¼} \angle \frac{60^o + K\ 360^o}{4}$$

$$\omega_k = \sqrt{2} \angle \frac{60^o + K\ 360^o}{4}$$

Taking $K = 0, 1, 2, 3$, in turn, gives the roots

$$\omega_o = \sqrt{2} \angle \frac{60^o}{4} = \sqrt{2} \angle 15^o$$

$$\omega_1 = \sqrt{2} \angle \frac{60^o + 360^o}{4} = \sqrt{2} \angle 105^o$$

$$\omega_2 = \sqrt{2} \angle \frac{60^o + 720^o}{4} = \sqrt{2} \angle 195^o$$

$$\omega_3 = \sqrt{2} \angle \frac{60^o + 1090^o}{4} = \sqrt{2} \angle 285^o$$

7.9 Principle Roots:

The n distinct nth roots of $x + jy$ are equal spaced about the circumference of a circle of radius $r^{1/n} = (\sqrt{x^2 + y^2})^{1/n}$, the smallest argument(for $K = 0$) being $\frac{\theta}{n}$. The root that is obtained by using this smallest value of θ is called the Principle root.

i.e.,   The principle root is: $Z^{1/n} = r^{1/n} \angle \frac{\theta}{n}$

Example 11: Solve the equation, $x^3 + j4 = 4\sqrt{3}$.

Solution:

$$x^3 = 4\sqrt{3} - 4j,\text{convert into polar form.}$$

$$x^3 = 8\angle - 30^o$$

$$x^3 = 8\angle 330^o$$

$$x = (8\angle 330^o)^{1/3}$$

$$= 2 \angle \frac{330^o + K\ 360^o}{3}$$

$$x_o = 2\angle 110^o$$

$$x_1 = 2\angle 230^o$$

$$x_2 = 2\angle 350^o$$

Hence, the solution set is: $2\angle 110^o,\ 2\angle 230^o, 2\angle 350^o$

7.10 Derivation of Euler's Relation:

By calculus Techniques, it can be shown that the sine and cosine functions of any angle x (in radian) are given by an expression containing an infinite series of terms.

$$\text{Sin } x = x - \frac{x^3}{3!} + \frac{x^3}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$$

(1)

$$\text{Cos } x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots\ldots\ldots \qquad (2)$$

By the same kind of Technique, e^x can also be expressed as an infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots\ldots\ldots \qquad (3)$$

Substitute $x = j\theta$,   we get,

$$e^{j\theta} = 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \ldots\ldots\ldots$$

And substituting values for power of j

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots\ldots\ldots \qquad (3)$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\ldots\ldots\ldots\right)$$

From equations (1) and (2)

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Exercise 7

Q.1:   If $A = 200 + j\,425$ and $B = 150 - j\,275$, find

   (a) $A + B$ (b) $A - 2B$

Q.2:   Express the following in the forms a + jb and $r \angle \theta$

   (a) $(5 + j4)^2$ (b) $(7-j2)\,(4 + j5)+ (3 - j)(5 - j2)$

   (c) $\dfrac{2 + j}{2 - j} + \dfrac{2 - j}{2 + j}$ (d) $\dfrac{(3+j3)(5-j3)}{3-j4}$

Q.3:   Express each of the following in Rectangular form i.e., a + jb form.

   (a) $3 \angle 45^o$ (b) $86 \angle - 115^o$

   (c) $2 \angle \dfrac{\pi}{6}$ rad   (d) $3 \angle 0^o$

Q.4:   Evaluate the following expressions:

   (a) $(5 \angle 45°)(3 \angle 36°)$ (b) $(1.1 \angle 30°)\,(2.3 \angle 17°)\,(2.8 \angle 74°)$

   (c) $\dfrac{3.7 \angle 17^o}{6.5 \angle 48^o}$ (d) $\dfrac{(8.7 \angle 76°)\,(6.8 \angle 62°)(1.2 \angle - 67°)}{(8.9 \angle 74°)\,(1.9 \angle 24°)}$

Q.5:   Evaluate the following expressions:

   (a) $(1 + j)^3$ (b) $(-2 + 3j)^4$

   (c) $\left[\left(2 \angle \dfrac{\pi}{4}\right)\left(3 \angle \dfrac{\pi}{4}\right)\right]^2$ (d) $\left(\dfrac{2\sqrt{2} \angle 30^o}{\sqrt{3} \angle 15^o}\right)^2$

Q.6:   Express the following in a + jb form.

   (a) $5e^{j\theta}$ (b) $10e^{j60^o}$ (c) $5\,e^{j\pi/3}$

Q.7:   Find the indicated roots of the following:

(a) $\sqrt{5 + j8}$ (b)   fifth roots of $-\sqrt{3} - j$

Q.8:   Find all the n of the nth roots of the following:

   (a) $Z = 32 \angle 45^o;\ n = 5$ (b) $Z = -16\sqrt{3} +j16 \ ; n =5$

Q.9:   Given $\omega = u + jv$ and $Z = x + jy$ and $\omega = 3Z^2$, express u and v in terms of x and y.

Q.10: The resultant impedance Z of two parallel circuits is given

   by $\dfrac{1}{Z} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$.

If $Z_1 = 5 - j3$ and $Z_2 = 3 + j5$ express Z in the form a + jb.

Q.12: In alternating current theory, the voltage V, current I and impedance Z may all be complex numbers and the basic relation between V, I, and Z is $I = \dfrac{V}{Z}$. Find a+ jb form.

(i)   I   When V= 10,   Z = 4 + j3

(ii)   V   When I = 3 + j8,   Z = 10 + j5

(iii)   Z   When I = 8 + j5 ,   V =20 – j8

Q.13: Solve the following equations:

(a) $x^2 = -j16$ (b) $x^5 = 16 - j16\sqrt{3}$

(Hints: find all n of the nth roots).

Answers 7

Q.1:   (a)   350 + j15   (b)   -100 + j975

Q.2:   (a)   9 + j40 ; 41 $\angle$ 77°18′

(b)   51 + j16;   53.46$\angle$ 17°25′

(c)   1.2 + j0; 1.2 $\angle$ 0   (d) $\dfrac{1}{25}$ (59 + j87); 4.288$\angle$ 55°51′

Q.3:   (a) $\dfrac{3}{\sqrt{2}} + j\dfrac{3}{\sqrt{2}}$ (b)   - 36.3 – j78

(c) $\sqrt{3} + j$ (d)   3 + oj

Q.4:   (a)   15 $\angle$ 81°   (b)   7.08 $\angle$ 121°

(c)   5.69 $\angle$ - 31°   (d)   4.34 $\angle$ -27°

Q.5:   (a) $2\sqrt{2} \angle 135°$ (b)   169 $\angle$ 134° 40′

(c)   36 $\angle$ 180°   (d)   2.667 $\angle$ 30°

Q.6:   (a)   5 + j0   (b)   5 - j8.66   (c)   2.5 + j4.33

Q.7:   (a)   3.907 $\angle$ 29°   ,   3.907 $\angle$ 209°

(b) $2^{1/5} \angle(42° + K72°)$,   k = 0, 1, …….4

Q.8:   (a)   2 $\angle$ 9° ; 2 $\angle$ 81°; 2 $\angle$ 153°; 2 $\angle$ 225° ; 2 $\angle$ 297°

(b) $2 \angle 30°$; $2 \angle 102°$; $2 \angle 174°$; $2 \angle 246°$; $2 \angle 318°$

Q.9: $U = 3 (x^2 - y^2)$, $V = 6xy$

Q.10: $Z = 4 + j$

Q.12: (i) $1.6 - j1.2$ (ii) $-10 + j95$ (iii) $1.348 - j1.843$

Q.13: (a) $4 \angle 135°$; $4 \angle 315°$ or $-2.83 + j2.83$; $2.83 - j2.83$

(b) $2 \angle 60°$; $2 \angle 132°$; $2 \angle 204°$; $2 \angle 276°$; $2 \angle 348°$

Short Questions

Write the short answers of the following:

Q1. Write the phasor (vector) $Z = a + jb$ in Trigonometric and Exponential form.

Q2. Express $\sqrt{2} \angle 45^0$ in Rectangular form. (i.e., $a + j b$)

Q3. Express $\sqrt{3} + j$ in Polar form.

Q4. Express $Z = e^{j\,\pi/3}$ in Rectangular form. (i.e., $a + j b$)

Q5. Find the product of $Z_1 = 2\angle 15^0$, $Z_2 = -1\angle 30^0$

Q6. Given that $Z_1 = 4\angle 60^0$ and $Z_2 = 2\angle 30^0$ find $\dfrac{Z_1}{Z_2}$

Q7. If $A = 2 + j\,3$ and $B = 8 + j\,5$, then find $A + B$

Q8. Simplify $(2 + j\,3)\,(4 - j\,2)$.

Q9. If $A = 20\angle 60^0$ and $B = 5\angle 30^0$, then find AB.

Q10. Simplify $\dfrac{(5\angle 45^0)(6\angle 60^0)}{3\angle 30^0}$

Q11. Write the conjugate and modulus of $-2 + j$.

Q12. Write the conjugate and modules of $\dfrac{-2}{3} - j\dfrac{4}{9}$

Simplify the Phasor(vector) and write the result in Rectangular form.

Q13. $(7\text{-}j2) - (4 + j5)$ **Q14.** $(-5+j3)\,(2 - j3)$

Q15. $\dfrac{-9+j4}{8-j3}$ **Q16.** $\dfrac{1}{4-j5} - \dfrac{1}{5-j4}$

Answers

Q1. $Z = r\,(\cos\theta + j\,\sin\theta)$, $Z = r\,e^{j\theta}$ **Q2.** $1 + j$ **Q3.**
$2(\cos 30^0 + j\,\sin 30^0)$ **Q4.** $1 + j\sqrt{3}$ **Q5.** $-\sqrt{2} + j\sqrt{2}$

Q6. $\dfrac{1}{2}\angle 30^0$ **Q7.** $10 + j\,8$ **Q8.** $14 + j\,8$

Q9. $100\angle 90^0$ **Q10.** $10\angle 75^0$ Q11. $-2 - j,\ 5\sqrt{5}$

Q12. $-\dfrac{2}{3} + j\dfrac{4}{9}\,,\ \dfrac{\sqrt{52}}{9}$ **Q13.** $3 - j7$ Q14. $-1 + j21$

Q15. $-\dfrac{84}{73} + j\dfrac{5}{73}$ **Q16.** $-\dfrac{1}{41} - j\dfrac{1}{41}$

OBJECTIVE TYPE QUESTIONS

Q.1   **Each questions has four possible answers. Choose the correct answer and encircle it.**

__1.   The value of $j^3 E$ is:

(a)   jE   (b)   -JE   (c)   E   (d)   -E

__2.   The trigonometric form of $Z = a + jb$ is:

(a) $|Z|, (\cos\theta - j\sin\theta)$ (b) $|Z| (\cos\theta + j\sin\theta)$

(c) $(\cos\theta + \sin\theta)$ (d) $(\cos\theta - j\sin\theta)$

__3.   The trigonometric form of $e^{-j\theta}$ is:

(a) $(\cos\theta + j\sin\theta)$ (b) $(\cos\theta - j\sin\theta)$

(c) $(\cos\theta + \sin\theta)$ (d) $(\cos\theta - \sin\theta)$

__4.   If $2(\cos 60° - j\sin 60°)$ then he exponential form is:

(a) $2e^{j60°}$ (b) $2 e^{-j60°}$ (c) $2e^{60°}$ (d) $2 e^{-60°}$

__5.   If $E (\cos\theta + j\sin\theta)$ then the exponential form is:

(a) $Ee^{j\theta}$ (b) $Ee^{-j\theta}$ (c) $e^{j\theta}$ (d) $e^{-j\theta}$

__6.   The trigonometric form of $1 + j\sqrt{3}$ is:

(a) $2(\cos 60° -- j\sin 60°)$ (b) $2(\cos 30° - j\sin 30°)$

(c) $2(\cos 60° + j\sin 60°)$ (d) $2(\cos 30° + j\sin 30°)$

__7.   The polar form of $-1- j$ is:

(a) $2 \angle 225°$ (b) $-2 \angle 225°$
(c) $-\sqrt{2} \angle 225°$ (d) $\sqrt{2} \angle 225°$

__8. $a + jb$ form of $2 \angle \dfrac{\pi}{6}$ is:

(a) $1 + j\sqrt{3}$ (b) $1 - j\sqrt{3}$
(c) $\sqrt{3} + j$ (d) $\sqrt{3} - j$

__9.   If $Z = 1 - j$, then arg Z is:
(a) $45°$ (b) $135°$ (c) $225°$ (d) $315°$

__10.   If $Z_1 = 8 \angle - 30°$ and $Z_2 = 2 \angle - 60°$, then $\dfrac{Z_1}{Z_2}$ is:

(a) $4 \angle 30°$ (b) $4 \angle - 30°$
(c) $4 \angle 90°$ (d) $4 \angle -90°$

Answers

1.	b	2.	b	3.	b	4.	a	5.	a
6.	c	7.	d	8.	c	9.	d	10.	a