Chapter – 8
COMPLEX NUMBERS

7.1 Introduction:

The real number system had limitations that were at first accepted and later overcome by a series of improvements in both concepts and mechanics. In connection with, quadratic, equations we encountered the concept of imaginary number and the device invented for handling it, the notation $i^2 = -1$or $i = \sqrt{-1}$. In this chapter we continue the extension of the real number system to include imaginary' numbers. The extended system is called the complex number system.

7.2 Complex Number:

A complex number is a number of the form $a + bi$, where a and b are real and $i^2 = -1$ or $i = \sqrt{-1}$. The letter 'a' is called the real part and 'b' is called the imaginary part of $a + bi$. If $a = 0$, the number ib is said to be a purely imaginary number and if $b = 0$, the number a is real. Hence, real numbers and pure imaginary numbers are special cases of complex numbers. The complex numbers are denoted by Z , i.e.,

$$Z = a + bi.$$

In coordinate form, $Z = (a, b)$.

Note : Every real number is a complex number with 0 as its imaginary part.

7.3 Properties of Complex Number:

(i) The two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$ for example if.

$$x - 2 + 4yi = 3 + 12\,i$$

Then $x - 2 = 3$ and $y = 3$

(ii) If any complex number vanishes then its real and imaginary parts will separately vanish.

For, if $a + ib = 0$, then

$$a = -ib$$

Squaring both sides

$$a^2 = -b^2$$

$$a^2 + b^2 = 0$$

Which is possible only when $a = 0$, $b = 0$

7.4 Basic Algebraic Operation on Complex Numbers:

There are four algebraic operations on complex numbers.

(i) **Addition:**

If $Z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2 i$, then

$$Z_1 + Z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$
$$= (a_1 + a_2) + i(b_1 + b_2)$$

(ii) **Subtraction:**

$$Z_1 - Z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

(iii) **Multiplication:**

$$Z_1 \; Z_2 = (a_1 + b_1 i) \, . \, (a_2 + b_2 i)$$
$$= a_1 \, a_2 + b_1 b i^2 + a_1 b_2 i + b_1 a_2 i$$
$$= (a_1 \, a_2 - b_1 b_2) + i(\, a_1 b_2 + b_1 a_2)$$

(iv) **Division:**

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

Multiply Numerator and denominator by the number $a_2 - b_2\, i$ in order to make the denominator real.

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \times \frac{a_2 - b_2 i}{a_2 - b_2 i}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 \, b_2)}{a_2^{\,2} + b_2^{\,2}}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \; \frac{b_1 a_2 - b_1 b_2}{a_2^{\,2} + b_2^{\,2}}$$

Generally result will be expressed in the form $a + ib$.

Example 1: Add and subtract the numbers $3 + 4i$ and $2 - 7i$.

Solution:

Addition: $(3 + 4i) + (2 - 7i) = (3 + 2) + i(4 - 7) = 5 - 3i$

Subtraction: $(3 + 4i) - (2 - 7i) = (3 - 2) + i(4 + 7) = 1 + 11i$

Example 2: Find the product of the complex numbers: $3 + 4i$ and $2 - 7i$.

Solution: $(3 + 4i)(2 - 7i) = 6 - 21i + 8i - 28i^2$

$$= 6 + 28 - 13i$$

$$= 34 - 13i$$

Example 3: Divide $3 + 4i$ by $2 - 7i$.

Solution: $\dfrac{3 + 4i}{2 - 7i} = \dfrac{3 + 4i}{2 - 7i} \times \dfrac{2 + 7i}{2 + 7i}$

$$= \frac{6 + 28i^2 + i(21 + 8)}{4 + 49}$$

$$= \frac{-22 + 29i}{53}$$

$$= \frac{-22}{53} + i\,\frac{29}{53}$$

Example 4: Express $\dfrac{(2 + i)(1 - i)}{4 - 3i}$ in the form of $a + ib$.

Solution: $\dfrac{(2 + i)(1 - i)}{4 - 3i} = \dfrac{(2 + 1) + i(1 - 2)}{4 - 3i} = \dfrac{3 - i}{4 - 3i}$

$$= \frac{3 - i}{4 - 3i} \times \frac{4 + 3i}{4 + 3i} = \frac{(12 + 3) + i(9 - 4)}{16 + 9}$$

$$= \frac{15 + i(5)}{25} = \frac{15}{25} + \frac{5}{25}\,i$$

$$= \frac{3}{5} + \frac{1}{5}\,i$$

Example 5: Separate into real and imaginary parts: $\dfrac{1 + 4i}{3 + i}$.

Solution :

$$\frac{1 + 4i}{3 + i} = \frac{1 + 4i}{3 + i} \times \frac{3 - i}{3 - i}$$

$$= \frac{(3 + 4) + i(12 - 1)}{9 + 1} = \frac{7 + 11i}{10}$$

$$= \frac{7}{10} + \frac{11}{10} i = \frac{7}{10} + \frac{11}{10} i$$

Here, real part = a = $\frac{7}{10}$

And imaginary part = b = $\frac{11}{10}$

Extraction of square roots of a complex number:

Example 6: Extract the square root of the complex numbers 21 – 20i.

Solution:

Let $a + ib = \sqrt{21 - 20i}$

Squaring both sides

$$(a + ib)^2 = 21 - 20\,i$$

$$a^2 - b^2 + 2abi = 21 - 20\,i$$

Comparing both sides

$$a^2 - b^2 = 21 \quad \text{.........................(1)}$$

$$2ab = -20 \quad \text{...........................(2)}$$

From (2) $b = -\frac{10}{a}$ Put b in equation (1),

$$a^2 - \frac{100}{a^2} = 21$$

$$a^4 - 21a^2 - 100 = 0$$

$$(a^2 - 25)(a^2 + 4) = 0$$

$$a^2 = 25 \quad \text{or} \quad a^2 = -4$$

$$a = \pm 5 \quad \text{or} \quad a = \pm\sqrt{-4} = \pm 2i$$

But a is not imaginary, so the real value of a is

$$a = 5 \quad \text{or} \quad a = -5$$

The corresponding value of b is

$b = -2 \quad \text{or} \quad b = 2$

Hence the square roots of $21 - 20i$ are:

$5 - 2i \quad \text{and} \quad -5 + 2i$

Factorization of a complex numbers:

Example 7: Factorise: $a^2 + b^2$

Solution:

We have
$$\begin{aligned} a^2 + b^2 &= a^2 -(- b^2) \\ &= a^2 -(i^2 b^2) \quad , \quad i^2 = -1 \\ &= (a)^2 - (ib)^2 \\ &= (a + ib)(a - ib) \end{aligned}$$

7.5 Additive Inverse of a Complex Number:

Let $Z = a + ib$ be a complex number, then the number $-a - ib$ is called the additive inverse of Z. It is denoted by $-Z$ i.e.,

$-Z = -a - ib.$ Also $Z - Z = 0$

Example 8: Find the additive inverse of 2 – 5i

Solution: Let $Z = 2 - 5i$

Then additive inverse of Z is:

$-Z = -(2 - 5i) = -2 + 5i.$

7.6 Multiplicative inverse of a complex number:

Let $a + ib$ be a complex number, then $x + iy$ is said to be multiplicative inverse of $a + ib$ if

$$(x + iy)(a + ib) = 1$$

Or
$$x + iy = \frac{1}{a + ib}$$
$$= \frac{1}{a + ib} \times \frac{a - ib}{a - ib}$$
$$x + iy = \frac{a - ib}{a^2 + b^2}$$
$$x + iy = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

So $x = \frac{a}{a^2 + b^2}$ $y = -\frac{b}{a^2 + b^2}$

Hence multiplicative inverse of (a, b) is $\left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right)$

Example 9: Find the multiplicative inverse of 4 + 3i or (4, 3).

Solution:

The multiplicative inverse of 4 + 3i is: $\frac{1}{4 + 3i}$

Since, $\frac{1}{4 + 3i} = \frac{1}{4 + 3i} \times \frac{3 - 3i}{4 - 3i}$

$$= \frac{4 - 3i}{16 + 9} = \frac{4}{25} - \frac{3}{25} i = \left(\frac{4}{25}, -\frac{3}{25}\right)$$

7.7 Conjugate of a complex number:

Two complex numbers are called the conjugates of each other if their real parts are equal and their imaginary parts differ only in sign.

If Z = a + bi, the complex number a – bi is called the conjugate of Z. it is denoted by $\bar{Z}$.

i.e., $\bar{Z} = \overline{a + bi} = a - bi$

Moreover $Z.\bar{Z} = (a + bi)(a - bi) = a^2 + b^2$

$$Z + \bar{Z} = 2a \quad \text{and} \quad Z - \bar{Z} = 2\,bi$$

Theorem: If Z_1 and Z_2 are complex numbers, then

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 \,.\, \bar{z}_2$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Proof:

Let $Z_1 = a_1 + bi$ and $Z_2 = a_2 + b_2 i$

(i) $Z_1 + Z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$

$$\overline{z_1 + z_2} = (a_1 + a_2) - i\,(b_1 + b_2)$$
$$= (a_1 - ib_1) + (a_2 - ib_2)$$
$$= \overline{z_1} + \overline{z_2}$$

(ii) $Z_1 Z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$$= (a_1 a_2 - b_1 b_2) + i\,((a_1 b_2 + b_1 b_2)$$

$$\overline{z_1 . z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2)$$
$$= a_1 a_2 - i a_1 b_2 - i b_1 a_2 - b_1 b_2$$
$$= a_1 (a_2 - ib_2) - ib_1 a_2 + i^2 b_1 b_2$$
$$= a_1 (a_1 - ib_2) - ib\,(a_2 - ib_2)$$
$$= a_1(a_2 - ib_2) - ib_1(a_2 - ib_2)$$
$$= (a_1 - ib_1)(a_2 - ib_2)$$
$$= \overline{z_1}\ .\ \overline{z_2}$$

(iii) $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$

$$= \frac{(a_1 + ib_1)\,(a_2 - ib_2)}{a_2^2 + b_2^2}$$

$$\frac{z_1}{z_2} = = \frac{(a_1 a_2 + b_1 b_2)}{a_2^2 + b_2^2} + i\,\frac{(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{(a_1 + ib_1)\,(a_2 - ib_2)}}{a_2^2 + b_2^2} = \frac{\overline{(a_1 + ib_1\,)}\,\overline{(a_2 - ib_2\,)}}{a_2^2 + b_2^2} \text{ by (ii)}$$

$$= \frac{(a_1 - ib_1)\,(a_2 + ib_2)}{(a_2 - ib_2)\,(a_2 + ib_2)} = \frac{a_1 - ib_1}{a_2 - ib_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Example 10: Evaluate $(3 + 4i)\,(\overline{3 + 4i})$

Solution:

$$(3 + 4i)\,(\overline{3 + 4i}) = (3 + 4i)\,(3 - 4i) = 9 + 16 = 25$$

Example 11: Find the conjugate of the complex number $2i(-3 + 8i)$

Solution:

$$2i\,(-3 + 8i) = -16 - 6i$$

$$\overline{2i(-3 + 8i)} = -16 + 6i$$

Alternately $\overline{2i(-3 + 8i)} = \overline{2i}\;\overline{(-3 + 8i)}$

By above theorem $= 2i\,(-3 - 8i)$

$= -16 + 6i$

Exercise 7.1

Q.1: Write each ordered pair (complex number) in the form: a + bi.

(i) (2, 6) (ii) (5, – 2)

(iii) (– 7, – 3) (iv) (4, 0)

Q.2: Write each complex number as an ordered pair.

(i) (2 + 3i) (ii) (- 3 + i) (iii) (4i) (iv)(0)

Q.3: Find the value of x and y in each of the following:

(i) $x + 3i + 3 = 5 + yi$ (ii) $x + 2yi = ix + y + 1$

(iii) $(x, y)\,(1, 2) = (-1, 8)$ (iv) $(x, -y)\,(3, -4) = (3, -29)$

(v) $(2x - 3y) + i(x - y)\,6 = 2 - i\,(2x - y + 3)$

Q.4: Simplify the following:

(i) $(2 - 3i) + (1 + 2i)$ (ii) $(3 + 5i) - (5 - 3i)$

(iii) $(9 + 7i) - (-9 + 7i) + (-18 + i)$ (iv) $(2 - 3i)(3 + 5i)$

(v) $(4 - 3i)^2$ (vi) $(3 + 4i)\,(4 + 3i)\,(2 - 5i)$

(vii) $(-1 + i\sqrt{3}\,)^3$ (viii) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-3}$

(ix) $\frac{1}{8 + 3i} + \frac{1}{8 - 3i}$ (x) $\frac{2 + i}{1 - 3i}$

(xi) $\frac{(3 + 4i)\,(1 - 2i)}{1 + i}$ (xii) $\frac{2 + \sqrt{-1}}{3 - \sqrt{-4}}$

(xiii) $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)^3$

Q.5: Find the conjugate of each of the following:

(i) $-2 + 3i$ (ii) $(1 + i)(-2 - i)$

(iii) $-3i(2 + 5i)$ (iv) $(-5 + 3i)\,(2 - 3i)$

Q.6: Reduce of the following to the form a + bi.

(i) $\frac{3 - i}{3 + 2i}$ (ii) $\frac{(2 + 3i)\,(3 + 2i)}{4 - 3i}$

(iii) $(2 - 3i)^2\,(\,\overline{3 + 4i}\,)$ (iv) $(\,\overline{4 + i}\,)\,(\,\overline{-1 + 3i}\,)$

(v) $(1 - i)^2\,(1 + i)$ (vi) $\frac{\sqrt{-3}}{1 - \sqrt{-7}}$

(vii) $(2 + \sqrt{-3}\,)\,(2 - \sqrt{-3}\,)$

Q.7: Factorize the following:

(i) $4m^2 + 9n^2$ (ii) $49a^2 + 625b^2$

(iii) $\frac{a^2}{9} + \frac{b^2}{81}$

Q.8: Find the multiplicative inverse of the following:

(i) $(-3, 4)$ (ii) $(\sqrt{2}\,, -\sqrt{5}\,)$

(iii) $\dfrac{2}{1+\sqrt{-1}}$ (iv) $-6-3i$

(v) $4-\sqrt{-7}$

Q.9: Extract the square root of the following complex numbers:

(i) $-3+4i$ (ii) $8-6i$

(iii) $24+10\,i$

Q.10: Prove that : $\dfrac{1}{\text{Cos}\theta - i\,\text{Sin}\theta} = \text{Cos}\,\theta + i\,\text{Sin}\,\theta$

Q.11: Express $x^2+y^2=a^2$ in terms of conjugate co-ordinates.

Q.12: The resultant impedance Z of two parallel circuits of impedances z_1 and z_2 is given by the formula $\dfrac{1}{z}=\dfrac{1}{z_1}+\dfrac{1}{z_2}$, Find the resultant impedance z when

$z_1 = 3+4i$, $z_2 = 4-2i$

Answers 7.1

(1) (i) $2+6i$ (ii) $5-2i$ (iii) $-7-3i$ (iv) 4

(2) (i) $(2, 3)$ (ii) $(-3, 1)$ (iii) $(0,4)$ (iv) $(0,0)$

(3) (i) $x=2, y=3$ (ii) $x=2, y=1$ (iii) $x=3, y=2$

(iv) $x=5, y=3$ (v) $x=-\dfrac{23}{10}, y=-\dfrac{11}{5}$

(4) (i) $3+i$ (ii) $-2+8i$ (iii) i

(iv) $21+i$ (v) $7-24i$ (vi) $125+50\,i$

(vii) 8 (viii) 1 (ix) $\dfrac{16}{73}$

(x) $-\dfrac{1}{10}+\dfrac{7}{10}i$ (xi) $\dfrac{(9-13i)}{2}$ (xii) $\dfrac{4}{13}+\dfrac{7}{13}i$

(xiii) 1

(5) (i) $-2-3i$ (ii) $-1+3i$ (iii) $15+6i$

(6) (i) $\dfrac{(7-9i)}{13}$ (ii) $\dfrac{(-39+52i)}{25}$ (iii) $-6-17i$

(iv) $-7-11i$ (v) $2-2i$ (vi) $\dfrac{-\sqrt{21}}{8}+\dfrac{\sqrt{3}}{8}i$

(vii) 7

(7) (i) $(2m + 3ni)(2m - 3ni)$ (ii) $(7a+25bi)(7a - 25bi)$

(iii) $\left(\frac{a}{3} + \frac{ib}{9}\right)\left(\frac{a}{3} - \frac{ib}{9}\right)$ or $\frac{1}{9}\left(a + \frac{ib}{3}\right)$

(8) (i) $\left(-\frac{3}{25}, -\frac{4}{25}\right)$ (ii) $\left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$ (iii) $\frac{1}{2} + \frac{1}{2}i$

(iv) $-\frac{6}{25} + i\frac{3}{45}$ (v) $\frac{4}{23} + \frac{\sqrt{7}}{23}i$

(9) (i) $\pm(1 + 2i)$ (ii) $\pm(3 - i)$ (iii) $\pm(5 + i)$

(11) $z\bar{z} = a^2$ (12) $z = 3.018 + 0.566i$

7.8 Graphical Representation:

Since a complex number Z = a + ib can also be represented by an ordered pair (a, b), each point in the plane can be viewed as the graph of a complex number. Thus, the graph of the complex number

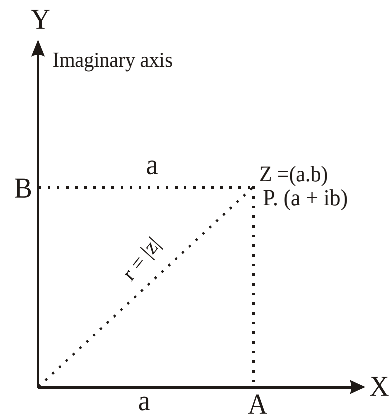

Fig.1

(a, b) or a + ib, is as shown in fig. 1. Since the real part a of a + ib taken as the x-coordinate of P, in this context the x-axis is called the real axis. Similarly, since the imaginary part b of a + ib is taken as y-coordinate of P, the y-axis is called the imaginary axis.

A plane on which complex numbers are thus represented is often called a complex plane. It is also sometimes called an Argand or Gauss Plane, after the French Mathematician Jean Robert Argand and the great German Mathemation Carl Friedrich Gauss.

7.9 Modulus of a Complex Number:

The Modulus or the absolute value of the complex number Z = a + ib is denoted by r, |Z| or |a + ib| and is given by,

$$r = |Z| = |a + ib| = \sqrt{a^2 + b^2}$$

Thus the modulus |a + ib| is just the distance from the origin to the point (a , b)

Note:

Since $z\bar{z} = a^2 + b^2 = |z|^2$, so $|z| = \sqrt{z\bar{z}}$

Example 11: Find The modulus of (3, – 5) and – 7 – i

Solution:

$$|(3, -5)| = |3 - 5i| = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25}$$

$$= \sqrt{34}$$

And $$|-7 - i| = \sqrt{(-7)^2 + (-1)^2} = \sqrt{49 + 1}$$

$$= \sqrt{50}$$

Theorem:

For complex numbers z_1 and z_2

(i) $|z_1 . z_2| = |z_1|\,|z_2|$

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(iii) $|z_1 + z_2| \le |z_1| + |z_2|$

(iv) $|z_1 - z_2| \ge |z_1| - |z_2|$

Proof:

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

(i)
$$|z_1 z_2| = |(a_1 + ib_1)(a_2 + ib_2)|$$
$$= |(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$
$$= \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2}$$
$$= \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2a_2^2}$$
$$= \sqrt{a_1^2(a_2^2 + b_2^2) + b_1^2(a_2^2 + b_2^2)}$$
$$= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$
$$= \sqrt{a_1^2 + b_1^2} \,.\, \sqrt{a_2^2 + b_2^2}$$
$$|z_1 z_2| = |z_1|\,|z_2|$$

(ii)
$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$$

$$= \frac{(a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}\right)^2 + \left(\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}\right)^2}$$

$$= \sqrt{\frac{a_1^2\,a_2^2 + b_1^2\,b_2^2 + b_1^2\,a_2^2 + a_1^2\,b_2^2}{(a_2^2 + b_2^2)^2}}$$

$$= \sqrt{\frac{(a_1^2 + b_1^2)\,(a_2^2 + b_2^2)}{(a_2^2 + b_2^2)^2}}$$

$$= \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}} = \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

(iii) $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

$$|z_1 + z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

$$= \sqrt{a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2}$$

$$= \sqrt{(a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2(a_1a_2 + b_1b_2)}$$

Squaring both sides:

$$|z_1 + z_2|^2 = (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2\,(a_1a_2 + b_1b_2)$$

$$= |z_1|^2 + |z_2|^2 + 2\sqrt{(a_1a_2 + b_1b_2)^2}$$

$$= |z_1|^2 + |z_2|^2 + 2\sqrt{a_1^2\,a_2^2 + b_1^2b_2^2 + 2a_1a_2\,b_1b_2}$$

as $2a_1\,a_2\,b_1\,b_2 \le a_1^2b_2^2 + b_1^2a_2^2$

So $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2$

$$+ 2\sqrt{a_1^2\,a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2a_2^2}$$

$$\le |z_1|^2 + |z_2|^2 + 2\sqrt{(a_1^2 + b_1^2)\,(a_2^2 + b_2^2)^2}$$

$$\le |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$$

$$|z_1 + z_2|^2 \le \left[|z_1| + |z_2|\right]^2$$

So $|z_1 + z_2| \leq |z_1| + |z_2|$

(iv) $|z_1| = |(z_1 - z_2) + z_2|$

$\leq |z_1 - z_2| + |z_2|$ by (iii)

$|z_1| - |z_2| \leq |z_1 - z_2|$

Or $|z_1 - z_2| \geq |z_1| - |z_2|$

7.10 Polar form of a complex number

In the fig. 2, we join the point P with the origin, we obtain the line r and the angle θ. Then the numbers or order pair (r, θ) are called the polar coordinates of the point P to distinguish them from the rectangular co-ordinates (x, y).

We call r the absolute value or modulus of z and θ, the angle from the positive real axis to this line, as the argument or amplitude of z and is denoted by arg z i.e., θ = arg z.

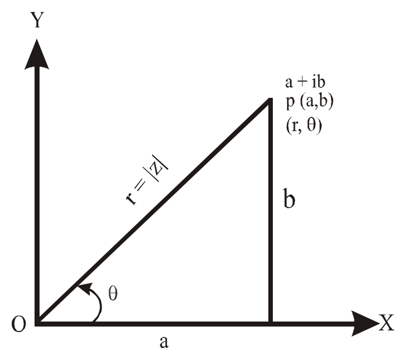

Fig.2

By use of Pythagorean theorem we have

$$\text{Cos } \theta = \frac{a}{r} \quad , \quad \text{Sin}\theta = \frac{b}{r}$$

$$a = r \text{ Cos } \theta \quad , \quad b = r \text{ Sin } \theta$$

$$r = |z| = \sqrt{a^2 + b^2} \quad ; \quad r \geq 0$$

and, $\tan \theta = \dfrac{b}{a}$

$$\theta = \arg Z = \tan^{-1}\left(\frac{b}{a}\right)$$

Therefore, the complex number

$$Z = a + ib$$

$$= r \text{ Cos } \theta + ir \text{ Sin } \theta$$

$$Z = r(\text{Cos } \theta + i \text{ Sin } \theta) \ldots\ldots\ldots\ldots\ldots..(1)$$

This is sometimes written as $Z = r \text{ Cis } \theta$

The right hand side of equation (1) is called the Trigonometric or polar form of Z.

The argZ has any one of an infinite number of real values differing by integral multiple $2k\pi$, where $k = 0, \pm 1, \pm 2, \ldots\ldots\ldots$ The values satisfying the relation. $\pi \leq \theta \leq \pi$ is called the principle value of the arg Z, denoted by Arg. Z.

Thus $\arg Z = \text{Arg } Z + 2k\pi$

Example 12: Express the complex number $1 + i\sqrt{3}$ in polar form.

Solution:

Here, we have $a = 1,\ b = \sqrt{3}$

$$r = \sqrt{a^2 + b^2} = \sqrt{1+3} = 2$$

and $\tan\theta = \dfrac{b}{a} = \sqrt{3}$

$$\theta = \tan^{-1}\sqrt{3} = 60^o$$

Hence $1 + i\sqrt{3} = 2\,[\text{Cos } 60^o + i \sin 60^o] = 2 \text{ Cis } 60^o$

Example13: Write $4(\text{Cos}225^o + i\,\text{Sin}225^o)$ in rectangular form.

Solution:

Since $\text{Cos } 225^o = \dfrac{-1}{\sqrt{2}}$ and $\text{Sin } 225^o = \dfrac{-1}{\sqrt{3}}$

So, $4\,(\text{Cos } 225^o + i \text{ Sin } 225^o) = 4\left(-\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}\right)$

$$= -\frac{4}{\sqrt{2}} - i\frac{4}{\sqrt{2}}$$

$$= -2\sqrt{2} - 2\sqrt{2}\,i$$

Alternately, $a = r \text{ Cos } \theta = 4\left(-\dfrac{1}{\sqrt{2}}\right) = -2\sqrt{2}$

And $b = r \text{ Sin } \theta = 4\left(-\dfrac{1}{\sqrt{2}}\right) = -2\sqrt{2}$

$$a + ib = -2\sqrt{2} - 2\sqrt{2}\ i$$

Example 14: Find the magnitude (modulus) and argument of $(4 + 7i)(3 - 2i)$

Solution:

$$(4 + 7i)(3 - 2i) = (12 + 14) + i(21 - 8)$$
$$= 26 + 13i$$
$$r = \sqrt{26^2 + 13^2} = \sqrt{676 + 169} = 29.1$$
$$\tan\theta = \frac{13}{26} = \frac{1}{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26^o\ 34'$$

Example 15: Find z such that $|Z| = \sqrt{2}$ and arg $Z = \frac{\pi}{4}$.

Solution:

Since, $|Z| = \sqrt{2}$, $\theta = \arg z = \frac{\pi}{4}$

$$a = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$$

and $b = r\sin\theta = = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$

$$Z = a + bi = 1 + i$$

Example 16: Show that the equation $|z + i| = 4$ represents a circle and find its centre and radius.

Solution:

$$|Z + i| = 4$$

Or $|x + iy + i| + I = 4$

Or $|x + i(y + 1)| = 4$

Or $\sqrt{x^2 + (y + 1)^2} = 4$

Or $x^2 + (y + 1)^2 = 16$

Which represents the circle with centre at (0, -1) and radius 4.

Exercise 7.2

Q.1: Find the magnitude (Modulus) of the following:

(i) -2 (ii) $3 + 2i$ (iii) $5i$ (iv) $(2, 0)$

(v) $(-2, 1)$ (vi) $(-2,-1)$ (vii) $\dfrac{1 + 2i}{2 - i}$

(viii) $\dfrac{(3 - 5i)(1 + i)}{4 + 2i}$

Q.2: Express of the following complex number in the polar (Trigonometric) form:

(i) $2 + 2\sqrt{3}\, i$ (ii) $1 - i$ (iii) $-1 - i$

(iv) $-1 + i\sqrt{3}$ (v) $-\sqrt{3} + i$ (vi) $-3 - 4i$

(vi) $8 - 15i$ (viii) -2 (ix) $2\sqrt{3} - 2i$

(x) $\left(\dfrac{2 + i}{3 - i}\right)^2$ (xi) $\dfrac{1 + 2i}{1 - 3i}$

Q.3: Write each complex number in the form a + bi

(i) $4 \text{ Cis } 240^o$ (ii) $3 \text{ Cis } 300^o$

(iii) $6 \text{ Cis } (-30^o)$ (iv) $12\text{Cis } 420^o$

Q.4: Find the magnitude and the principle argument of:

(i) $5 - 7i$ (ii) $8 + 5i$ (iii) $(5 - 7i)(8 + 5i)$

(iv) $\dfrac{5 - 7i}{8 + 5i}$ (v) $\dfrac{1 + i}{1 + i}$

Q.5: Find z such that:

(i) $|Z| = 8\sqrt{2}$, $\arg Z = \dfrac{\pi}{4}$ (ii) $|Z| = 5$, $\arg Z = -\dfrac{\pi}{2}$

(iii) $|Z| = 2$, $\arg Z = \dfrac{\pi}{6}$ (iv) $|Z| = \sqrt{6}$, $\arg Z = -\dfrac{\pi}{3}$

(v) $|Z| = \dfrac{1}{3}$, $\arg Z = \dfrac{\pi}{3}$ (vi) $|Z| = \sqrt{3}$, $\arg Z = -\pi$

Answers 7.2

Q.1: (i) 2 (ii) $\sqrt{13}$ (iii) 5 (iv) 2

(v) $\sqrt{5}$ (vi) $\sqrt{5}$ (vii) 1 (viii) $\sqrt{\frac{17}{5}}$

Q.2: (i) $4 \text{ Cis } 60^o$ (ii) $\sqrt{2} \text{ Cis } 315^o$ (iii) $\sqrt{2} \text{ Cis } 225^o$

(iv) $2 \text{ Cis } 120^o$ (v) $2 \text{ Cis } 150^o$ (vi) $5 \text{ Cis } 233^o\ 10'$

(vii) $17 \text{ Cis } 298^o$ (viii) $2 \text{ Cis } \pi$ (ix) $4 \text{ Cis } 330^o$

(x) $\frac{1}{2} \text{Cis} \frac{\pi}{2}$ (xi) $\frac{1}{\sqrt{2}} \text{ Cis } 135^o$

Q.3: (i) $-2 -2\sqrt{3}\, i$ (ii) $\frac{3}{2} - \frac{3\sqrt{3}}{2} i$ (iii) $3\sqrt{3} - 3i$

(iv) $6 + 6\sqrt{3}\, i$

Q.4: (i) $8.602, -54^o\ 28'$ (ii) $9.434, 32^o$

(iii) $81.16, -22^o\ 28'$ (iv) $0.912, -86^o\ 28'$

(vi) $1, 90^o$

Q.5: (i) $8 + 8i$ (ii) $-5i$ (iii) $\sqrt{3} + i$

(iv) $\frac{\sqrt{3}}{\sqrt{2}} - \frac{3}{\sqrt{2}} i$ (v) $\frac{1}{6} + \frac{\sqrt{3}}{6} i$ (vi) $-\sqrt{3}$

7.11 Multiplication and Division of Complex Numbers in Polar Form:

Generally addition and subtraction are better dealt with using Cartesian forms, but the multiplication and division of complex numbers can be found quite easily when the complex numbers are written in Trigonometric or polar form.

Multiplication:

Let $Z_1 = r_1 (\text{Cos } \theta_1 + i \text{ Sin } \theta_1)$

$Z_2 = r_2 (\text{Cos } \theta_1 + i \text{ Sin } \theta_1)$

$Z_1 . Z_2 = [r_1(\text{Cos } \theta_1 + i \text{ Sin } \theta_1)]\ [r_2(\text{Cos } \theta_2 + i \text{ Sin } \theta_2)]$

$=r_1r_2\,[(\text{Cos}\,\theta_1\,\text{Cos}\,\theta_2 - \text{Sin}\,\theta_1\,\text{Sin}\,\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$Z_1\,Z_2 = r_1r_2[\text{Cos}\,(\theta_1+\theta_2) + i\,\text{Sin}\,(\theta_1+\theta_2)]$

Or $Z_1\,Z_2 = r_1r_2\,\text{Cis}\,(\theta_1+\theta_2)$

Hence the absolute value of the product of two complex numbers is the product of their absolute values and the argument of the product of two complex numbers is the sum of their arguments.

Note: Since the product of two complex numbers is itself a complex number, we may find the product of any number of complex numbers by repeated application of this theorem.

i.e. $Z_1\,Z_1 = r_1r_1[\text{Cos}\,(\theta_1+\theta_1) + i\,\text{Sin}\,(\theta_1+\theta_1)]$

$Z_1^{\,2} = [r_1^{\,2}(\text{Cos}\,(2\theta_1) + i\,\text{Sin}\,(2\theta_1))$

Or $Z_1^{\,2} = r_1^{\,2}\,\text{Cis}\,2\theta_1$

Similarly $Z^3 = r^3\,\text{Cis}\,3\theta,$ for $Z = r\,(\text{Cos}\,\theta + i\,\text{Sin}\,\theta)$

A repeated application of this process, we get,

$$Z^n = r^n\,(\text{Cos}\,n\,\theta + i\,\text{Sin}\,n\,\theta) = r^n\,\text{Cis}\,n\,\theta$$

Similarly it $Z^{1/n}$ is the n-th root of Z, then put n = 1/n in above equation. then,

$$Z^{1/n} = r^{\,1/n}\,(\text{Cos}\,\frac{\theta}{n} + i\,\text{Sin}\,\frac{\theta}{n}) = r^{1/n}\,\text{Cis}\,\frac{\theta}{n}$$

Example 17: Multiply 4(Cis 30^o + i Sin 30^o) by 2 (Cis 60^o + i Sin 60^o)

Solution:

Let $Z_1 = 4\,(\,\text{Cos}\,30^o + i\,\text{Sin}\,30^o)$

And $Z_2 = 2\,(\,\text{Cos}\,60^o + i\,\text{Sin}\,60^o)$

Then $Z_{1.}\,Z_2 = 4\times 2[\text{Cos}\,(30^o+60^o) + i\,\text{Sin}\,(30^o+60^o)]$

$Z_{1.}\,Z_2 = 8\,(\text{Cis}\,90^o + i\,\text{Sin}\,90^o) = 8\,\text{Cis}\,90^o$

Example 18: Write $\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^3$ in the form a + bi

Solution: Let , $Z = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$, first we write it in polar form

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1, \qquad \theta = \tan^{-1}\left(\frac{\frac{-\sqrt{3}}{2}}{-\frac{1}{2}}\right)$$

$$\theta = \tan^{-1}\sqrt{3}$$

$$\theta = 240^o$$

$$Z = r\,[\text{Cos}\,\theta + i\,\text{Sin}\,\theta) = 1\ (\text{Cos}\,240^o + i\,\text{Sin}\,240^o)$$

$$Z = \left(-\frac{1}{3} - \frac{\sqrt{3}}{2}\right)^3 = 1^3\,[(\text{Cos}\,3\,(240^o) + i\,\text{Sin}\,3\,(240^o)]$$

$$= \text{Cos}\,720^o + i\,\text{Sin}\,720^o = 1$$

$$\frac{Z_1}{Z_2} = \frac{r_1(\text{Cos}\,\theta_1 + i\,\text{Sin}\,\theta_1)}{r_2\,(\text{Cos}\,\theta_2 + i\,\text{Sin}\,\theta_2)}$$

$$= \frac{r_1(\text{Cos}\,\theta_1 + i\,\text{Sin}\,\theta_1)}{r_2\,(\text{Cos}\,\theta_2 + i\,\text{Sin}\,\theta_2)} \times \frac{(\text{Cos}\,\theta_2 - i\,\text{Sin}\,\theta_2)}{(\text{Cos}\,\theta_2 - i\,\text{Sin}\,\theta_2)}$$

$$= \frac{r_1\left[(\text{Cos}\,\theta_1\cos\theta_2 + \text{Sin}\,\theta_1\,\text{Sin}\,\theta_2) + i(\text{Sin}\,\theta_1\,\text{Cos}\,\theta_2 - \cos\theta_1\text{Sin}\,\theta_2)\right]}{r_2(\text{Cos}^2\,\theta_2 + \text{Sin}^2\,\theta_2)}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\left[\,\text{Cos}\,(\theta_1 - \theta_2) + i\,\text{Sin}(\theta_1 - \theta_2)\right]$$

Or $$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\,\text{Cis}\,(\theta_1 - \theta_2)$$

Hence the absolute value of the quotient of two complex numbers is the quotient of their absolute values and the argument of the quotient is the angle of the dividend minus the angle of the divisor.

Example 19:

$$\frac{8\,\text{Cis}\,540^o}{2\,\text{Cis}\,225^o} = 4\,\text{Cis}\,(540^o - 225^o)$$

$$= 4\,\text{Cis}\,(315^o)$$

$$= 4\,(\text{Cos}\,315^o + i\,\text{Sin}\,315^o)$$

Since $\text{Cos}\,315^o = \frac{1}{\sqrt{2}}$ and $\text{Sin}\,315^o = -\frac{1}{\sqrt{2}}$

So $$\frac{8\,\text{Cis}\,540^o}{2\,\text{Cis}\,225^o} = 4\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right)$$

$$= 2\sqrt{2} - 2\sqrt{2}\, i$$

Example 20: Find the quotient of $1 + \sqrt{3}\, i$ and $1 + i$.

Solution:

We first write each of them in polar form.

Let $Z_1 = 1 + \sqrt{3}\, i$ and $Z_2 = 1 + i$

$r_1 = \sqrt{1 + 3} = 2$ and $r_2 = \sqrt{1 + 1} = \sqrt{2}$

$\theta_1 = \tan^{-1}\sqrt{3} = 60^o$ and $\theta_2 = \tan^{-1}(1) = 45^o$

So, $Z_1 = 2(\text{Cos } 60^o + i \text{ Sin } 60^o)$ and $Z_2 = \sqrt{2}(\text{Cos } 45^o + i \text{ Sin } 45^o)$

Now,
$$\frac{z_1}{z_2} = \frac{2(\text{Cos } 60^o + i \text{ Sin } 60^o)}{\sqrt{2}(\text{Cos } 45^o + i \text{ Sin } 45^o)}$$
$$= \sqrt{2}\,[\text{Cos } (60^o - 45^o) + i \text{ Sin } (60^o - 45^o)]$$
$$= \sqrt{2}\,[\text{Cos } 15^o + i \text{ Sin } 15^o] = \sqrt{2} \text{ Cis } 15^o$$

Exercise 7.3

Q.1: Perform the indicated operations in each and express the results in the form a + ib.

(i) $[3(\text{Cos } 22^o + i \text{ Sin } 22^o]\,[2(\text{Cos } 8^o + i \text{ Sin } 8^o]$

(ii) $[4\,(\text{ Cos } 29^o + \text{I Sin } 29^o)]\left[\frac{1}{2}(\text{Cos } 16^o + i \text{ Sin } 16^o)\right]$

(iii) $(\sqrt{5} \text{ Cis } 28^o)\,(\sqrt{5} \text{ Cis } 8^o)\,(2 \text{ Cis } 9^o)$

(iv) $\dfrac{6(\text{Cos } 51^o + i \text{ Sin } 51^o)}{2(\text{Cos } 21^o + i \text{ Sin } 21^o)}$

(v) $\dfrac{10(\text{Cos } 143^o + i \text{ Sin } 143^o)}{5(\text{Cos } 8^o + i \text{ Sin } 8^o)}$

(vi) $\dfrac{(\text{Cis } 180^o)\,(6 \text{ Cis } 99^o)}{3\,(\text{Cos } 39^o + i \text{ Sin } 39^o)}$

(vii) $\dfrac{15(\text{Cos } 48^o + i \text{ Sin } 48^o)}{(3 \text{ Cis } 46^o)\,(2 \text{ Cis } 32^o)}$

Q.2: Perform the indicated operations and give the results in polar form.

(i) $(1 + i)(1 - \sqrt{3}\, i)$ (ii) $(-\sqrt{3} - i)(-1 + i)$

(iii) $(1 + i)^4$ (iv) $(1 - i\sqrt{3})^2$

(v) $\sqrt{\dfrac{1 + i}{1 - i}}$ (vi) $\dfrac{-1 - i}{-1 + i}$

(vii) $\dfrac{(1 + i\sqrt{3})(\sqrt{3} + i)}{1 + i}$

Q.3: Show that $\left|\dfrac{1 + 2i}{2 - i}\right| = 1$

Answers 7.3

Q.1: (i) $3(\sqrt{3} + i)$ (ii) $\sqrt{2}(1 + i)$ (iii) $5\sqrt{2}(1 + i)$

(iv) $1.5(\sqrt{3} + i)$ (v) $\sqrt{2}(-1 + i)$ (vi) $-5(1 + i\sqrt{3})$

(vii) $1.25(\sqrt{3} - i)$

Q.2: (i) $2\sqrt{2}$ Cis 345^o (ii) $2\sqrt{2}$ Cis 345^o

(iii) Cos 180^o (iv) 4 Cis 240^o

(v) Cis 45^o (vi) Cis 90^o

(vii) $2\sqrt{2}$ Cis 45^o

Short Questions

Write the short answers of the following:

Q.1: Write the conjugate and modulus of- 2 + *i*

Q.2: Write the conjugate and modulus of $\frac{-2}{3} - \frac{4}{9}i$

Q.3: Simplify the complex numbers:

(i)$(2 +5 i) +(- 3 +i)$ (ii)$(7-2i) - (4 +5i)$ (iii) $(-5+3i)(2 - 3i)$

(iv) $\frac{-9 + 4i}{8 - 3i}$

Q.4: Find the addition inverse of (3, - 8).

Q.5: Find the conjugate and modulus of $\frac{1 + i}{1 - i}$

Q.6: Prove that if $z = \bar{z}$, then $\bar{z}$ is real.

Q.7: Find the values of x and y from the equation.

$(2x - y - 1) - i(x - 3y) = (y - x) - i (2 - 2y)$

Q.8: Show that $\left|\frac{1 +2 i}{2 - i}\right| = 1$

Q.9: If $z = 2 + 3 i$ Prove that $z\bar{z} = 13$

Q.10: Find the multiplication inverse of (-3, 4)

Q.11: Find the conjugate and modulus of complex number $\frac{1 +2 i}{2 - i}$

Q.12: Factorize $36a^2 + 100b^2$

Q.13: Factorize $2x^2 +5y^2$

Q.14: Factorize $9a^2 + 64b^2$

Q.15: Write complex number $-\sqrt{2} + \sqrt{6}\, i$ in polar (trigonometric)form.

Q.16: Express complex number $3 - \sqrt{3}\, i$ in polar(trigonometric)form.

Express the following complex number in the form x + *i* y

Q.17: When |Z| = 6 and arg Ƶ $= \frac{3\overline{\wedge}}{4}$

Q.18: When |Z| = 3 and arg Ƶ $= -\frac{\pi}{2}$

Q.19: Express |Z| = 2, arg Ƶ $= \frac{\pi}{3}$

Q.20: Show that $Ƶ^2 + \overline{Ƶ}^2$ is a real number

<u>Answers</u>

Q1. $-2 - i$, $\sqrt{5}$ **Q2.** $-\frac{2}{3} + \frac{4}{9}i$, $\frac{\sqrt{52}}{9}$

Q3. (i) $1 + 6i$ (ii) $3 - 7i$ (iii) $-1 + 21i$ (iv) $-\frac{84}{73} + \frac{5}{73}i$

Q4. $-3 + 8i$ **Q5.** i , 1

Q7. $x = -5$, $y -7$ **Q10.** $-\frac{3}{25} - \frac{4}{25}i$ **Q11.** $-i$, 1

Q12. $(6a - 10bi)(6a + 10bi)$ **Q13.** $(\sqrt{2}\,x - \sqrt{5}\,yi)(\sqrt{2}\,x + \sqrt{5}\,yi)$

Q14. $(3a - 8bi)(3a + 8bi)$ **Q15.** $2\sqrt{2}\,[\cos 60^o - i \sin 60^o]$

Q16. $2\sqrt{3}\,[\cos 30^o - i \sin 30^o]$ **Q17.** $-3\sqrt{2} + 3\sqrt{2}\,i$

Q18. $0 - 3i$ **Q19.** $1 + i\sqrt{3}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. The additive inverse of a + ib is:

(a) -a + ib (b) a – ib (c) –a – ib (d) a + ib

__2. Sum of -3 + 5i and 4 - 7i is:

(a) 1 – 2i (b) -1 – 2i (c) 1-12i (d) -7 +12i

__3. Conjugate of (2 + 3i) + [1 – i] is:

(a) 3 - 2i (b) 3 + 4i (c) 3 – 4i (d) 3 + 2i

__4. Modulus of 3 + 4i is:

(a) 47 (b) 16 (c) 5 (d) 3

__5. Product of 2 + 3i and 2 – 3i is:

(a) $\sqrt{13}$ (b) 13 (c) $\sqrt{2}$ (d) $\sqrt{-5}$

__6. Ordered pair form of -3 - 2i is:

(a) (3, 2) (b) (-3, -2) (c) (-3, 2) (d) (3, -2)

__7. If z = a + ib then Z+ $\overline{Z}$ is equal to:

(a) 2a (b) 2b (c) 0 (d) 2a + 2ib

__8. (1 + 2i) (3 – 5i) is equal to:

(a) 13 + i (b) -2 - i (c) -4 + 3i (d) 3i

__9. i[2 – i] is equal to:

(a) 1 + 2i (b) $2i + i^2$ (c) 3 (d) 3i

__10. If z= a + ib then, $\overline{Z}$ is equal to:

(a) a + ib (b) a – ib (c) a + b (d) a – b

ANSWERS

1.	c	2.	a	3.	a	4,	c	5.	b
6.	b	7.	a	8.	a	9.	a	10.	b