Chapter 9
Partial Fractions

9.1 Introduction: A fraction is a symbol indicating the division of integers. For example, $\frac{13}{9}$, $\frac{2}{3}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator N(x) and the divisor (lower number) is called the denominator, D(x).

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

i) $\frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

ii) $\frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{9x^2+5x-3}{(x+1)^2(x-2)}$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions.**

9.2 Partial fractions :

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**.

For example,

$$\frac{2x + x^2 - 1}{x(x^2-1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

$\frac{2x + x^2 - 1}{x(x^2-1)}$ is the resultant fraction and $\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$ are its partial fractions.

9.3 Polynomial:

Any expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real constants, if $a_n \neq 0$ then P(x) is called polynomial of degree n.

9.4 Rational fraction:

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$, with no common factors, is called a rational fraction. A rational fraction is of two types:

9.5 Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of numerator N(x) is less than the degree of Denominator D(x).

For example

(i) $\frac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$

(ii) $\frac{6x + 27}{3x^3 - 9x}$

9.6 Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called an improper fraction if the degree of the Numerator N(x) is greater than or equal to the degree of the Denominator D(x)

For example

(i) $\frac{2x^3 - 5x^2 - 3x - 10}{x^2 - 1}$

(ii) $\frac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1}$

<u>Note:</u> An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction.

For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Which is obtained as, divide $6x^2 + 5x^2 - 7$ by $3x^2 - 2x - 1$ then we get a polynomial (2x+3) and a proper fraction $\frac{8x - 4}{x^2 - 2x - 1}$

9.7 Process of Finding Partial Fraction:

A proper fraction $\frac{N(x)}{D(x)}$ can be resolved into partial fractions as:

(I) If in the denominator D(x) a linear factor (ax + b) occurs and is non-repeating, its partial fraction will be of the form $\frac{A}{ax + b}$,where A is a constant whose value is to be determined.

(II) If in the denominator D(x) a linear factor (ax + b) occurs n times, i.e., $(ax + b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+.....+\frac{A_n}{(ax+b)^n}$$

,where A_1, A_2, A_3 - - - - - - - - - A_n are constants whose values are to be determined

(III) If in the denominator D(x) a quadratic factor $ax^2 + bx + c$ occurs and is non-repeating, its partial fraction will be of the form

$\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants whose values are to be determined.

(IV) If in the denominator a quadratic factor $ax^2 + bx + c$ occurs n times, i.e., $(ax^2 + bx + c)^n$,then there will be n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+$$

$$- - - - - - - - - - + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where A_1, A_2, A_3 - - - - - - - - A_n and B_1, B_2, B_3 - - - - - - - B_n are constants whose values are to be determined.

Note: The evaluation of the coefficients of the partial fractions is based on the following theorem:

If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal, for example , if

$px^2 + qx + a = 2x^2 - 3x + 5 \quad \forall\, x$, then

$p = 2, \quad q = -3$ and $a = 5$.

9.8 Type I

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x-25}{(x-3)(x-4)}$ into partial fractions.

Solution:

$$\frac{7x-25}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4} \text{ ------------------(1)}$$

Multiplying both sides by L.C.M. i.e., $(x - 3)(x - 4)$, we get

$7x - 25 = A(x - 4) + B(x - 3)$ -------------- (2)

$7x - 25 = Ax - 4A + Bx - 3B$

$7x - 25 = Ax + Bx - 4A - 3B$

$7x - 25 = (A + B)x - 4A - 3B$

Comparing the co-efficients of like powers of x on both sides, we have

$7 = A + B$ and
$-25 = -4A - 3B$

Solving these equation we get
$A = 4$ and $B = 3$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$7(4) - 25 = A(4 - 4) + B(4 - 3)$
$28 - 25 = 0 + B(1)$
$B = 3$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$7(3) - 25 = A(3 - 4) + B(3 - 3)$
$21 - 25 = A(-1) + 0$
$-4 = -A$
$A = 4$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Note : The R.H.S of equation (1) is the identity equation of L.H.S

Example 2:

write the identity equation of $\frac{7x-25}{(x-3)(x-4)}$

Solution : The identity equation of $\frac{7x-25}{(x-3)(x-4)}$ is

$$\frac{7x-25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Example 3:

Resolve into partial fraction: $\frac{1}{x^2 - 1}$

Solutios: $\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$

$$1 = A(x + 1) + B(x - 1) \qquad (1)$$

Put $x - 1 = 0, \Rightarrow x = 1$ in equation (1)

$1 = A(1 + 1) + B(1 - 1) \Rightarrow A = \frac{1}{2}$

Put $x + 1 = 0$, $\Rightarrow$ $x = -1$ in equation (1)

$1 = A(-1+1) + B(-1-1)$

$1 = -2B$, $\Rightarrow$ $B = \frac{1}{2}$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Example 4:

Resolve into partial fractions $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Solution:

This is an improper fraction first we convert it into a polynomial and a proper fraction by division.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x+3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Let $$\frac{8x - 4}{x^2 - 2x - 1} = \frac{8x - 4}{(3x+1)} = \frac{A}{x-1} + \frac{B}{3x+1}$$

Multiplying both sides by $(x - 1)(3x + 1)$ we get

$$8x - 4 = A(3x + 1) + B(x - 1) \qquad \text{(I)}$$

Put $x - 1 = 0$, $\Rightarrow$ $x = 1$ in (I), we get

The value of A

$$8(1) - 4 = A(3(1) + 1) + B(1 - 1)$$

$$8 - 4 = A(3 + 1) + 0$$

$$4 = 4A$$

$$\Rightarrow \qquad A = 1$$

Put $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$ in (I)

$$8\left(-\frac{1}{3}\right) - 4 = B\left(-\frac{1}{3} - 1\right)$$

$$-\frac{8}{3} - 4 = \left(-\frac{4}{3}\right)$$

$$-\frac{20}{3} = -\frac{4}{3} B$$

$$\Rightarrow \qquad B = \frac{20}{3} \text{ x } \frac{3}{4} = 5$$

Hence the required partial fractions are

$$\frac{6x^3+5x^2-7}{3x^2-2x-1} = (2x+3)+\frac{1}{x-1}+\frac{5}{3x+1}$$

Example 5:

Resolve into partial fraction $\dfrac{8x-8}{x^3-2x^2-8x}$

Solution: $$\frac{8x-8}{x^3-2x^2-8x} = \frac{8x-8}{x(x^2-2x-8)} = \frac{8x-8}{x(x-4)(x+2)}$$

Let $$\frac{8x-8}{x^3-2x^2-8x} = \frac{A}{x}+\frac{B}{x-4}+\frac{C}{x+2}$$

Multiplying both sides by L.C.M. i.e., $x(x-4)(x+2)$

$$8x-8 = A(x-4)(x+2)+Bx(x+2)+Cx(x-4) \qquad \text{(I)}$$

Put $x = 0$ in equation (I), we have

$8(0)-8 = A(0-4)(0+2)+B(0)(0+2)+C(0)(0-4)$

$-8 = -8A+0+0$

$\Rightarrow \quad A = 1$

Put $x-4 = 0 \quad \Rightarrow \quad x = 4$ in Equation (I), we have

$8(4)-8 = B(4)(4+2)$

$32-8 = 24B$

$24 = 24B$

$\Rightarrow \quad B = 1$

Put $x+2 = 0 \Rightarrow x = -2$ in Eq. (I), we have

$8(-2)-8 = C(-2)(-2-4)$

$-16-8 = C(-2)(-6)$

$-24 = 12C$

$\Rightarrow \quad C = -2$

Hence the required partial fractions

$$\frac{8x-8}{x^3-2x^2-8x} = \frac{1}{x}-\frac{1}{x-4}-\frac{2}{x+2}$$

Exercise 9.1

Resolve into partial fraction:

Q.1 $\dfrac{2x+3}{(x-2)(x+5)}$ Q.2 $\dfrac{2x+5}{x^2+5x+6}$

Q.3 $\dfrac{3x^2-2x-5}{(x-2)(x+2)(x+3)}$ Q.4 $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Q.5 $\dfrac{x}{(x-a)(x-b)(x-c)}$ Q.6 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Q.7 $\dfrac{2x^3-x^2+1}{(x+3)(x-1)(x+5)}$ Q.8 $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

Q.9 $\dfrac{6x+27}{4x^3-9x}$ Q.10 $\dfrac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$

Q.11 $\dfrac{x^4}{(x-1)(x-2)(x-3)}$ Q.12 $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

Answers 9.1

Q.1 $\dfrac{1}{x-2}+\dfrac{1}{x+5}$ Q.2 $\dfrac{1}{x+2}+\dfrac{1}{x+3}$

Q.3 $\dfrac{3}{20(x-2)}-\dfrac{11}{4(x-2)}+\dfrac{28}{5(x+3)}$

Q.4 $1+\dfrac{3}{x-4}-\dfrac{24}{x-5}+\dfrac{30}{x-6}$

Q.5 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-b)(c-a)(x-c)}$

Q.6 $\dfrac{a^2}{(a-b)(a-c)(1-ax)}+\dfrac{b^2}{(b-a)(b-c)(1-bx)}+\dfrac{c^2}{(c-b)(c-a)(1-cx)}$

Q.7 $2+\dfrac{31}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{137}{6(x+5)}$

Q.8 $\dfrac{1}{2(1-x)}-\dfrac{4}{(1-2x)}+\dfrac{9}{2(1-3x)}$

Q.9 $\dfrac{3}{x}+\dfrac{4}{2x-3}+\dfrac{2}{2x+3}$

Q.10 $\dfrac{2}{x-1}-\dfrac{3}{2x-1}+\dfrac{4}{x+12}$

Q.11 $x+6+\dfrac{1}{2(x-1)}-\dfrac{16}{x-2}+\dfrac{81}{2(x-3)}$

Q.12 $1 + \frac{1}{x} - \frac{1}{5(x-1)} - \frac{8}{5(2x+3)}$

9.9 Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\frac{x^2 - 3x + 1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x - 1)^2 (x - 2)$, we get

$x^2 - 3x + 1 = A(x - 1)(x - 2) + B(x - 2) + C(x - 1)^2$ (I)

Putting $x - 1 = 0 \Rightarrow x = 1$ in (I), then

$(1)^2 - 3(1) + 1 = B(1 - 2)$

$1 - 3 + 1 = -B$

$-1 = -B$

$\Rightarrow B = 1$

Putting $x - 2 = 0 \Rightarrow x = 2$ in (I), then

$(2)^2 - 3(2) + 1 = C(2 - 1)^2$

$4 - 6 + 1 = C(1)^2$

$\Rightarrow -1 = C$

Now $x^2 - 3x + 1 = A(x^2 - 3x + 2) + B(x - 2) + C(x^2 - 2x + 1)$

Comparing the co-efficient of like powers of x on both sides, we get

$A + C = 1$

$A = 1 - C$

$= 1 - (-1)$

$= 1 + 1 = 2$

$\Rightarrow A = 2$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

Example 2:

Resolve into partial fraction $\frac{1}{x^4(x+1)}$

Solution

$$\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

Where A, B, C, D and E are constants. To find these constants multiplying both sides by L.C.M. i.e., x^4 (x + 1), we get

$1 = A(x^3)(x + 1) + Bx^2 (x + 1) + Cx (x + 1) + D(x + 1) + Ex^4$ (I)

Putting $x = -1$ in Eq. (I)

$1 = E(-1)^4$

$\Rightarrow$ $E = 1$

Putting x = 0 in Eq. (I), we have

$1 = D(0 + 1)$

$1 = D$

$\Rightarrow$ $D = 1$

$1 = A(x^4 + x^3)+B(x^3 + x^2)+C(x^2 + x)+D(x + 1) + Ex$

Comparing the co-efficient of like powers of x on both sides.

Co-efficient of x^3 : $A + B = 0$ ……… (i)

Co-efficient of x^2 : $B + C = 0$ ……… (ii)

Co-efficient of x : $C + D = 0$ ……… (iii)

Putting the value of D = 1 in (iii)

$C + 1 = 0$

$\Rightarrow$ $C = -1$

Putting this value in (ii), we get

$B - 1 = 0$

$\Rightarrow$ $B = 1$

Putting B = 1 in (i), we have

$A + 1 = 0$

$\Rightarrow$ $A = -1$

Hence the required partial fraction are

$$\frac{1}{x^4(x+1)} = \frac{-1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}$$

Example 3:

Resolve into partial fractions $\dfrac{4+7x}{(2+3x)(1+x)^2}$

Solution:

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

Multiplying both sides by L.C.M. i.e., $(2 + 3x)(1 + x)^2$

We get $4 + 7x = A(1 + x)^2 + B(2 + 3x)(1 + x) + C(2 + 3x)$ …. (I)

Put $2 + 3x = 0 \quad \Rightarrow \quad x = -\frac{2}{3}$ in (I)

Then $4+7\left(-\frac{2}{3}\right) = A\left(1-\frac{2}{3}\right)^2$

$$4-\frac{14}{3} = A\left(-\frac{1}{3}\right)^2$$

$$-\frac{2}{3} = \frac{1}{9}A$$

$\Rightarrow \quad A = \frac{-2}{3} \times \frac{9}{1} = -6$

$A = -6$

Put $1 + x = 0 \quad \Rightarrow \quad x = -1$ in eq. (I), we get

$4 + 7(-1) = C(2 - 3)$

$4 - 7 = C(-1)$

$-3 = -C$

$\Rightarrow \quad C = 3$

$4 + 7x = A(x^2 + 2x + 1) + B(2 + 5x + 3x^2) + C(2 + 3x)$

Comparing the co-efficient of x^2 on both sides

$A + 3B = 0$

$-6 + 3B = 0$

$3B = 6$

$\Rightarrow \quad B = 2$

Hence the required partial fraction will be

$$\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

Exercise 4.2

Resolve into partial fraction:

Q.1 $\frac{x+4}{(x-2)^2(x+1)}$

Q2. $\frac{1}{(x+1)(x^2-1)}$

Q.3 $\frac{4x^3}{(x+1)^2(x^2-1)}$

Q.4 $\frac{2x+1}{(x+3)(x-1)(x+2)^2}$

Q.5 $\frac{6x^2-11x-32}{(x+6)(x+1)^2}$

Q.6 $\frac{x^2-x-3}{(x-1)^3}$

Q.7 $\dfrac{5x^2+36x-27}{x^4-6x^3+9x^2}$ Q.8 $\dfrac{4x^2-13x}{(x+3)(x-2)^2}$

Q.9 $\dfrac{x^4+1}{x^2(x-1)}$ Q.10 $\dfrac{x^3-8x^2+17x+1}{(x-3)^3}$

Q.11 $\dfrac{x^2}{(x-1)^3(x+2)}$ Q.12 $\dfrac{2x+1}{(x+2)(x-3)^2}$

Answers4.2

Q.1 $-\dfrac{1}{3(x-2)}+\dfrac{2}{(x-2)^2}+\dfrac{1}{3(x+1)}$

Q.2 $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}-\dfrac{1}{2(x+1)^2}$

Q.3 $\dfrac{1}{2(x-1)}+\dfrac{7}{2(x+1)}-\dfrac{5}{(x+1)^2}+\dfrac{2}{(x+1)^3}$

Q.4 $\dfrac{5}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{4}{3(x+2)}+\dfrac{1}{(x+2)^2}$

Q.5 $\dfrac{10}{x+6}-\dfrac{4}{x+1}-\dfrac{3}{(x-1)^2}$

Q.6 $\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{3}{(x-1)^3}$

Q.7 $\dfrac{2}{x}-\dfrac{3}{x^2}-\dfrac{2}{(x-3)}+\dfrac{14}{(x-3)^2}$

Q.8 $\dfrac{3}{x+3}+\dfrac{1}{x-2}-\dfrac{2}{(x-2)^2}$

Q.9 $x+1\ -\dfrac{1}{x}-\dfrac{1}{x^2}+\dfrac{2}{x-1}$

Q.10 $1+\dfrac{1}{x-3}-\dfrac{4}{(x-3)^2}+\dfrac{7}{(x-3)^3}$

Q.11 $\dfrac{4}{27(x-1)}+\dfrac{5}{9(x-1)^2}+\dfrac{1}{3(x-1)^3}-\dfrac{4}{27(x+2)}$

Q.12 $-\dfrac{3}{25(x+2)}+\dfrac{3}{25(x-3)}+\dfrac{7}{5(x-3)^2}$

9.10 Type III:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\dfrac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x + 3)(x^2 + 1)$, we get

$$9x - 7 = A(x^2 + 1) + (Bx + C)(x + 3) \qquad \text{(I)}$$

Put $x + 3 = 0 \quad \Rightarrow \quad x = -3$ in Eq. (I), we have

$9(-3) -7 = A((-3)^2 + 1) + (B(-3) + C)(-3 + 3)$

$-27 -7 = 10A + 0$

$A = -\dfrac{34}{10} \quad \Rightarrow \quad \boxed{A = -\dfrac{17}{5}}$

$$9x - 7 = A(x^2 + 1) + B(x^2 + 3x) + C(x + 3)$$

Comparing the co-efficient of like powers of x on both sides

$$A + B = 0$$
$$3B + C = 9$$

Putting value of A in Eq. (i)

$$-\frac{17}{5} + B = 0 \quad \Rightarrow \quad \boxed{B = \frac{17}{5}}$$

From Eq. (iii)

$$C = 9 - 3B \quad = 9 - 3\left(\frac{17}{4}\right)$$

$$= 9 - \frac{51}{5} \quad \Rightarrow \quad \boxed{C = -\frac{6}{5}}$$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

Example 2:

Resolve into partial fraction $\dfrac{x^2+1}{x^4+x^2+1}$

Solution:

Let $\dfrac{x^2+1}{x^4+x^2+1} = \dfrac{x^2+1}{(x^2-x+1)(x^2+x+1)}$

$$\frac{x^2+1}{(x^2-x+1)(x^2+x+1)} = \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2+x+1)}$$

Multiplying both sides by L.C.M. i.e., $(x^2-x+1)(x^2+x+1)$

$x^2 + 1 = (Ax + B)(x^2 + x + 1) + (Cx + D)(x^2 - x + 1)$

Comparing the co-efficient of like powers of x, we have

Co-efficient of x^3 : $A + C = 0$ …………. (i)

Co-efficient of x^2 : $A + B - C + D = 1$ ……….. (ii)

Co-efficient of x : $A + B + C - D = 0$ ……….. (iii)

Constant $B + D = 1$ ……….. (iv)

Subtract (iv) from (ii) we have

$A - C = 0$ …………… (v)

$A = C$ …………… (vi)

Adding (i) and (v), we have

$A = 0$

Putting A = 0 in (vi), we have

$C = 0$

Putting the value of A and C in (iii), we have

$B - D = 0$ …………… (vii)

Adding (iv) and (vii)

$2B = 1 \quad \Rightarrow \quad B = \dfrac{1}{2}$

from (vii) B = D, therefore

$D = \dfrac{1}{2}$

Hence the required partial fraction are

$$\frac{0x+\frac{1}{2}}{(x^2-x+1)} + \frac{0x+\frac{1}{2}}{(x^2+x+1)}$$

i.e., $\dfrac{1}{2(x^2-x+1)} + \dfrac{1}{2(x^2+x+1)}$

Exercise 4.3

Resolve into partial fraction:

Q.1 $\dfrac{x^2+3x-1}{(x-2)(x^2+5)}$ Q.2 $\dfrac{x^2-x+2}{(x+1)(x^2+3)}$

Q.3 $\dfrac{3x+7}{(x+3)(x^2+1)}$

Q.4 $\dfrac{1}{(x^3+1)}$

Q.5 $\dfrac{1}{(x+1)(x^2+1)}$

Q.6 $\dfrac{3x+7}{(x^2+x+1)(x^2-4)}$

Q.7 $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$

Q.8 $\dfrac{x+a}{x^2(x-a)(x^2+a^2)}$

Q.9 $\dfrac{x^5}{x^4-1}$

Q.10 $\dfrac{x^2+x+1}{(x^2-x-2)(x^2-2)}$

Q.11 $\dfrac{1}{x^3-1}$

Q.12 $\dfrac{x^2+3x+3}{(x^2-1)(x^2+4)}$

Answers 4.3

Q.1 $\dfrac{1}{x-2}+\dfrac{3}{x^2+5}$

Q.2 $\dfrac{1}{x+1}-\dfrac{1}{x^2+3}$

Q.3 $-\dfrac{1}{5(x+3)}+\dfrac{x+12}{5(x^2+1)}$

Q.4 $\dfrac{1}{3(x+1)}-\dfrac{(x-2)}{3(x^2-x+1)}$

Q.5 $\dfrac{1}{2(x+1)}-\dfrac{x-1}{2(x^2+1)}$

Q.6 $\dfrac{13}{28(X-2)}-\dfrac{1}{12(X+2)}-\dfrac{8X+31}{21(X^2+X+1)}$

Q.7 $\dfrac{1}{x+1}+\dfrac{2x-2}{x^2-x+3}$

Q.8 $\dfrac{1}{a^3}\left[\dfrac{1}{X-a}+\dfrac{x}{X^2+a^2}-\dfrac{2}{X}-\dfrac{a}{X^2}\right]$

Q.9 $x+\dfrac{1}{4(x-1)}+\dfrac{1}{4(x+1)}-\dfrac{x}{2(x^2+1)}$

Q.10 $\dfrac{1}{3(x+1)}+\dfrac{7}{6(x-2)}-\dfrac{3x+2}{2(x^2-2)}$

Q.11 $\dfrac{1}{3(x-1)}-\dfrac{x+2}{3(x^2+x+1)}$

Q.12 $\dfrac{7}{10(x-1)}-\dfrac{1}{10(x+1)}-\dfrac{3x-1}{5(x^2+4)}$

9.11 Type IV: Quadratic repeated factors

When the denominator has repeated Quadratic factors.

Example 1:

Resolve into partial fraction $\dfrac{x^2}{(1-x)(1+x^2)^2}$

Solution:

$$\frac{x^2}{(1-x)(1+x^2)^2} = \frac{A}{1-x} + \frac{Bx + C}{(1+x^2)} + \frac{Dx + E}{(1+x^2)^2}$$

Multiplying both sides by L.C.M. i.e., $(1-x)(1+x^2)^2$ on both sides, we have

$x^2 = A(1 + x^2)^2 + (Bx + C)(1 - x)(1 + x^2)+(Dx + E)(1 - x)$ ……(i)

$x^2 = A(1 + 2x^2 + x^4) + (Bx + C)(1 - x + x^2 - x^3)+(Dx + E)(1 - x)$

Put $1 - x = 0 \Rightarrow x = 1$ in eq. (i), we have

$(1)^2 = A(1 + (1)^2)^2$

$1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$

$x^2 = A(1 + 2x^2 + x^4) + B(x - x^2 + x^3 - x^4)+C(1 - x + x^2 - x^3)$
$+ D(x-x^2)+ E(1 - x)$ …………………… (ii)

Comparing the co-efficient of like powers of x on both sides in Equation (II), we have

Co-efficient of x^4	:	$A - B = 0$	…………….	(i)
Co-efficient of x^3	:	$B - C = 0$	…………….	(ii)
Co-efficient of x^2	:	$2A - B + C - D = 1$	…….	(iii)
Co-efficient of x	:	$B - C + D - E = 0$	…….	(iv)
Co-efficient term	:	$A + C + E = 0$	…………….	(v)

from (i), $B = A$

$\Rightarrow B = \frac{1}{4} \quad \because \quad A = \frac{1}{4}$

from (i) $B = C$

$\Rightarrow C = \frac{1}{4} \quad \because \quad C = \frac{1}{4}$

from (iii) $D = 2A - B + C - 1$

$$= 2\left(\frac{1}{4}\right) - \frac{1}{4} + \frac{1}{4} - 1$$

$\Rightarrow \boxed{D = -\frac{1}{2}}$

from (v) $E = -A - C$

$$E = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Hence the required partial fractions are by putting the values of A, B, C, D, E,

$$\frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{4}x + \frac{1}{4}}{1+x^2} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(1+x^2)^2}$$

$$\frac{1}{4(1-x)} + \frac{(x+1)}{4(1+x^2)} - \frac{x+1}{2(1+x^2)^2}$$

Example 2:

Resolve into partial fractions $\dfrac{x^2 + x + 2}{x^2(x^2+3)^2}$

Solution:

Let $\dfrac{x^2 + x + 2}{x^2(x^2+3)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx + D}{x^2+3} + \dfrac{Ex + F}{(x^2+3)^2}$

Multiplying both sides by L.C.M. i.e., $x^2(x^2+3)^2$, we have

$$x^2 + x + 2 = Ax(x^2+3)^2 + B(x^2+3)^2$$
$$+(cx + D)x^2(x^2+3) + (Ex + F)(x^2)$$

Putting x = 0 on both sides, we have

$2 = B\,(0 + 3)^2$
$2 = 9B \quad \Rightarrow$ $\boxed{B = \frac{2}{9}}$

Now $x^2 + x + 2 = Ax(x^4 + 6x^2 + 9) + B(x^4 + 6x^2 + 9)$
$$+C(x^5 + 3x^2) + D(x^4 + 3x^2) + E(x^3) + Fx^2$$

$$x^2 + x + 2 = (A + C)x^5 + (B + D)x^4 + (6A + 3C + E)x^3$$
$$+(6B + 3D + F)x^2 + (x + 9B)$$

Comparing the co-efficient of like powers of x on both sides of Eq. (I), we have

Co-efficient of x^5 : $A + C = 0$ ……… (i)

Co-efficient of x^4 : $B - D = 0$ ……… (ii)

Co-efficient of x^3 : $6A + 3C + E = 0$ ……… (iii)

Co-efficient of x^2 : $6B + 3D + F = 1$ ……… (iv)

Co-efficient of x : 9A = 1 ……… (v)

Co-efficient term : 9B = 1 ……… (vi)

from (v) 9A = 1

$$\Rightarrow \quad \boxed{A = \frac{1}{9}}$$

from (i) A + C = 0

C = –A

$$\Rightarrow \quad \boxed{C = -\frac{1}{9}}$$

from (i) B + D = 0

D = –B

$$\Rightarrow \quad \boxed{D = -\frac{2}{9}}$$

from (iii) 6A + 3C + E =

$$6\left(\frac{1}{9}\right) + 3\left(-\frac{1}{9}\right) + E = 0$$

$$E = \frac{3}{9} - \frac{6}{9}$$

$$\Rightarrow \quad \boxed{E = -\frac{1}{3}}$$

from (iv) 6B + 3D + F = 1

$$F = 1 - 6B - 3D$$

$$= 1 - 6\left(\frac{2}{9}\right) - 3\left(\frac{2}{9}\right)$$

$$= 1 - \frac{12}{9} + \frac{6}{9}$$

$$\Rightarrow \quad \boxed{F = \frac{1}{3}}$$

Hence the required partial fractions are

$$\frac{\frac{1}{9}}{x} + \frac{\frac{2}{9}}{x^2} + \frac{-\frac{1}{9}x - \frac{2}{9}}{x^2+3} + \frac{-\frac{1}{3}x + \frac{1}{3}}{(x^2+3)^2}$$

$$= \frac{1}{9x} + \frac{2}{9x^2} - \frac{x+2}{9(x^2+3)} - \frac{x-1}{3(x^2+3)^2}$$

Exercise 4.4

Resolve into Partial Fraction:

Q.1 $\dfrac{7}{(x+1)(x^2+2)^2}$

Q.2 $\dfrac{x^2}{(1+x)(1+x^2)^2}$

Q.3 $\dfrac{5x^2+3x+9}{x(x^2+3)^2}$

Q.4 $\dfrac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$

Q.5 $\dfrac{2x^4-3x^2-4x}{(x+1)(x^2+2)^2}$

Q.6 $\dfrac{x^3-15x^2-8x-7}{(2x-5)(1+x^2)^2}$

Q.7 $\dfrac{49}{(x-2)(x^2+3)^2}$

Q.8 $\dfrac{8x^2}{(1-x^2)(1+x^2)^2}$

Q.9 $\dfrac{x^4+x^3+2x^2-7}{(x+2)(x^2+x+1)^2}$

Q.10 $\dfrac{x^2+2}{(x^2+1)(x^2+4)^2}$

Q.11 $\dfrac{1}{x^4+x^2+1}$

Answers 4.4

Q.1 $\dfrac{7}{9(x+1)}-\dfrac{7x-7}{9(x^2+2)}-\dfrac{7x-7}{3(x^2+2)^2}$

Q.2 $\dfrac{1}{4(1+x)}-\dfrac{x-1}{4(1+x^2)}+\dfrac{x-1}{2(1+x^2)^2}$

Q.3 $\dfrac{1}{x}-\dfrac{x}{x^2+3}+\dfrac{2x+3}{(x^2+3)^2}$

Q.4 $\dfrac{2}{x-1}+\dfrac{2x-1}{x^2+x+1}+\dfrac{3}{(x^2+x+1)^2}$

Q.5 $\dfrac{1}{3(x+1)}+\dfrac{5(x-1)}{3(x^2+2)}-\dfrac{2(3x-1)}{(x^2+1)^2}$

Q.6 $-\dfrac{2}{2x-5}+\dfrac{x+3}{1+x^2}+\dfrac{x-2}{(1+x^2)^2}$

Q.7 $\dfrac{1}{x-2}-\dfrac{x+2}{x^2+3}-\dfrac{7x+14}{(x^2+3)^2}$

Q.8 $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}-\dfrac{4}{(1+x^2)^2}$

Q.9 $\dfrac{1}{x+2}+\dfrac{2x-3}{(x^2+x+1)^2}-\dfrac{1}{x^2+x+1}$

Q.10 $\dfrac{1}{9(x^2+1)}-\dfrac{1}{9(x^2+4)}+\dfrac{2}{3(x^2+4)^2}$

Q.11 $-\dfrac{(x-1)}{2(x^2-x+1)}+\dfrac{(x+1)}{2(x^2+x+1)}$

Summary

Let N(x) ≠ and D(x) ≠ 0 be two polynomials. The $\dfrac{N(x)}{D(x)}$ is called a proper fraction if the degree of N(x) is smaller than the degree of D(x).

For example: $\dfrac{x-1}{x^2+5x+6}$ is a proper fraction.

Also $\dfrac{N(x^1)}{D(x)}$ is called an improper fraction of the degree of N(x) is greater than or equal to the degree of D(x).

For example: $\dfrac{x^5}{x^4-1}$ is an improper fraction.

In such problems we divide N(x) by D(x) obtaining a quotient Q(x) and a remainder R(x) whose degree is smaller than that of D(x).

Thus $\dfrac{N(x)'}{D(x)}=Q(x)+\dfrac{R(x)}{D(x)}$ where $\dfrac{R(x)'}{D(x)}$ is proper fraction.

Types of proper fraction into partial fractions.

Type 1: Linear and distinct factors in the D(x)

$$\frac{x-a}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b}$$

Type 2: Linear repeated factors in D(x)

$$\frac{x-a}{(x+a)(x^2+b^2)}=\frac{A}{x+a}+\frac{Bx+C}{x^2+b^2}$$

Type 3: Quadratic Factors in the D(x)

$$\frac{x-a}{(x+a)(x^2+b)^2}=\frac{A}{x+a}+\frac{Bx+C}{x^2+b^2}$$

Type 4: Quadratic repeated factors in D(x):

$$\frac{x-a}{(x^2+a^2)(x^2+b^2)}=\frac{Ax+B}{x^2+a^2}+\frac{Cx+D}{x^2+b^2}+\frac{Ex+F}{(x^2+b^2)^2}$$

Short Questions:

Write the short answers of the following:

Q.1: What is partial fractions?

Q.2: Define proper fraction and give example.

Q.3: Define improper fraction and given one example:

Q.4: Resolve into partial fractions $\frac{2x}{(x - 2)(x + 5)}$

Q.5: Resolve into partial fractions: $\frac{1}{x^2 - x}$

Q.6: Resolve $\frac{7x + 25}{(x + 3)(x + 4)}$ into partial fraction.

Q.7: Resolve $\frac{1}{x^2 - 1}$ into partial fraction:

Q.8: Resolve $\frac{x^2 + 1}{(x + 1)(x - 1)}$ into partial fractions.

Q.9: Write an identity equation of $\frac{8x^2}{(1 - x^2)(1 + x^2)^2}$

Q.10: Write an identity equation of $\frac{2x + 5}{x^2 + 5x + 6}$

Q.11: Write identity equation of $\frac{x - 5}{(x + 1)(x^2 + 3)}$

Q.12: Write an identity equation of $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Q.13: Write an identity equation of $\frac{(x - 1)(x - 2)(x - 3)}{(x - 4)(x - 5)(x - 6)}$

Q.14: Write an identity equation of $\frac{x^5}{x^4 - 1}$

Q.15: Write an identity equation of $\frac{2x^4 - 3x^2 - 4x}{(x + 1)(x^2 + 2)^2}$

Q16. Form of partial fraction of $\frac{1}{(x + 1)(x - 2)}$ is ___________.

Q.17. Form of partial fraction of $\frac{1}{(x + 1)^2(x - 2)}$ is ___________.

Q.18. Form of partial fraction of $\frac{1}{(x^2 + 1)(x - 2)}$ is ___________.

Q.19. Form of partial fraction of $\frac{1}{(x^2 + 1)(x - 4)^2}$ is ___________.

Q.20. Form of partial fraction of $\dfrac{1}{(x^3-1)(x^2+1)}$ is __________.

Answers

Q4. $\dfrac{4}{7(x-2)} - \dfrac{10}{7(x+5)}$

Q5. $\dfrac{-1}{x} + \dfrac{1}{x-1}$

Q6. $\dfrac{4}{x+3} + \dfrac{3}{x+4}$

Q7. $\dfrac{1}{x^2-1} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

Q8. $1 + \dfrac{1}{x+1} + \dfrac{1}{x-1}$

Q9. $\dfrac{A}{1-x} + \dfrac{B}{1+x} + \dfrac{Cx+D}{1+x^2} + \dfrac{Ex+F}{(1+x^2)^2}$

Q10. $\dfrac{A}{x+2} + \dfrac{B}{x+3}$

Q11. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+3}$

Q12. $(2x+3) + \dfrac{A}{x-1} + \dfrac{B}{3x+1}$

Q13. $1 + \dfrac{A}{4-4} + \dfrac{B}{x-5} + \dfrac{C}{x-6}$

Q14. $x + \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$

Q15. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{(x^2+2)^2}$

Q16. $\dfrac{A}{x+1} + \dfrac{B}{x-2}$

Q17.

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Q18. $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$

Q19.

$$\frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Q20. $\dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+x+1)} + \dfrac{Dx+E}{x^2+1}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. If the degree of numerator N(x) is equal or greater than the degree of denominator D(x), then the fraction is:

(a) proper (b) improper
(c) Neither proper non-improper (d) Both proper and improper

_2. If the degree of numerator is less than the degree of denominator, then the fraction is:

(a) Proper (b) Improper
(c) Neither proper non-improper (d) Both proper and improper

__3. The fraction $\frac{2x+5}{x^2+5x+6}$ is known as:

(a) Proper (b) Improper
(c) Both proper and improper (d) None of these

__4. The number of partial fractions of $\frac{6x+27}{4x^3-9x}$ are:

(a) 2 (b) 3
(c) 4 (d) None of these

__5. The number of partial fractions of $\frac{x^3-3x^2+1}{(x-1)(x+1)(x^2-1)}$ are:

(a) 2 (b) 3
(c) 4 (d) 5

__6. The equivalent partial fraction of $\frac{x+11}{(x+1)(x-3)^2}$ is:

(a) $\frac{A}{x+1}+\frac{B}{(x-3)^2}$ (b) $\frac{A}{x+1}+\frac{B}{x-3}$

(c) $\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$ (d) $\frac{A}{x+1}+\frac{Bx+C}{(x-3)^2}$

__7. The equivalent partial fraction of $\frac{x^4}{(x^2+1)(x^2+3)}$ is:

(a) $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$ (b) $\frac{Ax+B}{x^2+1}+\frac{Cx}{x^2+3}$

(c) $1+\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$ (d) $\frac{Ax}{x^2+1}+\frac{Bx}{x^2+3}$

__8. Partial fraction of $\dfrac{2}{x(x+1)}$ is:

(a) $\dfrac{2}{x}-\dfrac{1}{x+1}$ (b) $\dfrac{1}{x}-\dfrac{2}{x+1}$

(c) $\dfrac{2}{x}-\dfrac{2}{x+1}$ (d) $\dfrac{2}{x}+\dfrac{2}{x+1}$

__9. Partial fraction of $\dfrac{2x+3}{(x-2)(x+5)}$ is called:

(a) $\dfrac{2}{x-2}+\dfrac{1}{x+5}$ (b) $\dfrac{3}{x-2}+\dfrac{1}{x+5}$

(c) $\dfrac{2}{x-2}+\dfrac{3}{x+5}$ (d) $\dfrac{1}{x-2}+\dfrac{1}{x+5}$

__10. The fraction $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$ is called:

(a) Proper (ii) Improper

(c) Both proper and Improper (iv) None of these

Answers:

1.	b	2.	a	3.	a	4.	b	5.	c
6.	c	7.	c	8.	c	9.	d	10.	B