# Chapter 13

# The Circle

## 13.1 Circle:

A circle is the set of all points in a plane that are equally distants from a fixed point. The fixed point is called the centre of circle and the distance from the centre to any point on the circle is called the radius of the circle.

An equation of a circle is an equation in x and y which is satisfied by the coordinates of a point if and only if the point is on the circle.

## 13.2 Standard Form of the Equation of a Circle:

Let P(x,y) be a point in a plane which moves so that it is always a constant distance, called the radius r, from the fixed point (h, k), called the centre of the circle. Then by distance formula

$(x - h)^2 + (y - k)^2 = r^2$……..…(1)

Equation (1) is **called standard form** of the circle, with centre (h, k) and radius r.

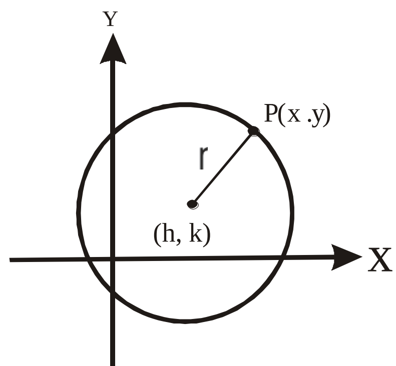


Figure 9.1

If the centre is at the origin (0, 0), equation (1) reduces to

$x^2 + y^2 = r^2$……………………..(2)

Also if the centre is at the origin (0,0) and radius is 1 (one),then the equation (1) reduces to the unit circle i.e;

$x^2 + y^2 = 1$

**Note** that any equation equivalent to equation (1) is also an equation of the circle. We may reduce the equation (1) to the form.

$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$ …………(3)

we observe that

(i) The equation (3) is second degree in x and y.

(ii) The coefficients of $x^2$ and $y^2$ are equal.

(iii) There is no product term xy.

**Example 1: Find the equation of the circle with centre at (–2, 3) and radius 6.**

**Solution:** From the standard form.

$(x-h)^2 + (y-k)^2 = r^2$, here (h, k) = (–2, 3) , r = 6

$(x+2)^2 + (x-3)^2 = 36$

**Example 2: Find the centre and radius of the circle $x^2 + y^2 + x - 4y - \frac{7}{4} = 0$ .**

**Solution:** $x^2 + y^2 + x - 4y - \frac{7}{4} = 0$

Add. $\left(\frac{1}{2}\right)^2 + (2)^2$ on both sides

$$x^2 + x + \left(\frac{1}{2}\right)^2 + y^2 - 4y + (2)^2 = \frac{7}{4} + \frac{1}{4} + 4$$

$$(x + \frac{1}{2})^2 + (y-2)^2 = 6$$

Comparing with $(x-h)^2 + (y-k)^2 = r^2$

Hence the centre is at $( -\frac{1}{2}, 2 )$ and the radius is $\sqrt{6}$

**Theorem: (General form of an equation of a circle)**

The equation $x^2 + y^2 + 2gx + 2fy + c + 0$………………..(4)

represents a circle with centre (–g, –f) and radius $\sqrt{g^2 + f^2 - c}$

**Proof :**

Since , $x^2 + y^2 + 2gx + 2fy + c = 0$

$x^2 + 2gx + y^2 + 2fy = -c$

Add $g^2 + f^2$ on both sides

$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$………………..(5)

Comparing this equation with the standard form

i.e., $(x - h)^2 + (y - k)^2 = r^2$

We have $h = -g$, $k = -f$, $r = \sqrt{g^2 + f^2 - c}$

Thus equation (4) represents a circle with

centre $(h, k) = (-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Equation (4) is **called the General form of the circle**

**Note:** From equation (5) we find

$r^2 = g^2 + f^2 - c$

(i) If $r^2 > 0$, the circle is real.

(ii) If $r^2 = 0 \Rightarrow r = 0$, the circle is a point circle

(iii) If $r^2 < 0$, the circle is an imaginary circle.

These are **called special features** of equation of circle.

**Example 3: What type of the circle is represented by $x^2 + y^2 + 2x - 4y + 8 = 0$.**

**Solution:** Here, $g = 1, f = -2, c = 8$

Since $r^2 = g^2 + f^2 - c$

$= 1 + (-2)^2 - 8$

$r^2 = -3 < 0$

Hence the equation represents an imaginary circle.

**Example 4: By comparing with the general form, find the centre and radius of the circle $2x^2 + 2y^2 - 5x + 4y - 7 = 0$.**

**Solution:** The given circle has the equation

$$x^2 + y^2 - \frac{5}{2}x + 2y - \frac{7}{2} = 0$$

Comparing it with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We have, $g = -\frac{5}{4}$, $f = 1$, $c = -\frac{7}{2}$

Hence the centre is $(-g, -f) = \left(\frac{5}{2}, -1\right)$

and radius $r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(\frac{5}{4}\right)^2 + (-1)^2 - \left(-\frac{7}{2}\right)}$$

$$= \sqrt{\frac{25}{16} + 1 + \frac{7}{2}} = \sqrt{\frac{97}{4}}$$

## 13.3 Circle Determined by Three Conditions:

From the general form of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We see that there are three effective constants g, f and c. In the general form three conditions can be imposed upon them which will determined a circle, unique or otherwise.

### 13.3.I Circle Through Three Points:

If we substitute the coordinates of any point, we get three linear equations, in the three unknowns, which may be solved simultaneously for g, f and c. By substituting these values in general form we get the equation of the circle.

### 13.3.2 Circles Tangent to Line:

Instead of specifying that the circle pass through certain points we may require that it is tangent to certain line or that its centre lie on a given line. Combinations of point and line conditions may be used to determine a circle (or circles).

**Example 5: Find an equation of the circle passing through (9,– 7), ( –3, –1) and (6, 2).**

**Solution:**

There are at least two ways of proceeding.

**Method –I:**

The General form of an equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

We must determine the constants g, f and c.

Since the circle through the three given points, so

Put point (9, –7) $\Rightarrow 9^2 + (-7)^2 + 2(9)\,g + 2(-7)f + c = 0$ (2)

Put point (–3,–1) $\Rightarrow (-3)^2 + (-1)^2 + 2(-3)g + 2\,(-1)f + c = 0$ (3)

Put point (6, 2) $\Rightarrow (6)^2 + (2)^2 + 2\,(6)g + 2(2)\,f + c = 0$ (4)

Equivalent equations are

$18\,g - 14f + c = -130$ (2′)

$-6g - 2f + c = -10$ (3′)

$12g + 4f + c = -40$ (4′)

Subtracting equation (4′) from (2′) and (3′) we get,

$6g - 18f = -90$ ------------------------------------ (5)

$-18g - 6\,f = 30$ ------------------------------------ (6)

Dividing equation (6) by 3, we get

$-6g - 2f = 10$ ------------------------------------ (7)

Adding equation (5) and (7), we get ,

$-20\,f = -80$

$\boxed{f = 4}$

put f = 4 in equation (7) $\Rightarrow -6g - 8 = 10$

$-6g = 18$

$\boxed{\mathbf{g = -3}}$

put values of f and g in equation (3′)

$18 - 8 + c = -10$

$\boxed{\mathbf{C = -20}}$

Substituting g, f and c in equation (1), we get

$x^2 + y^2 - 6x + 8y - 20 + 0$

This method has the disadvantage that we must do more work to find the centre and radius of the circle.

**Method –II:**

This method is more geometric. It depends on the geometric fact that the centre of the circle is at the intersection of the perpendicular bisectors of segments joining the three given points. It sufficent to use only two of these perpendicular bisectors.

As we see from Fig. 2, the slope of

$\ell$ is $\frac{-7+1}{9+3} = -\frac{1}{2}$

and the slope of

n is $\frac{2+7}{6-9} = -3$

The slope of the perpendicular bisector $\ell_\perp$ and $n_\perp$ are 2 and $\frac{1}{3}$ respectively.

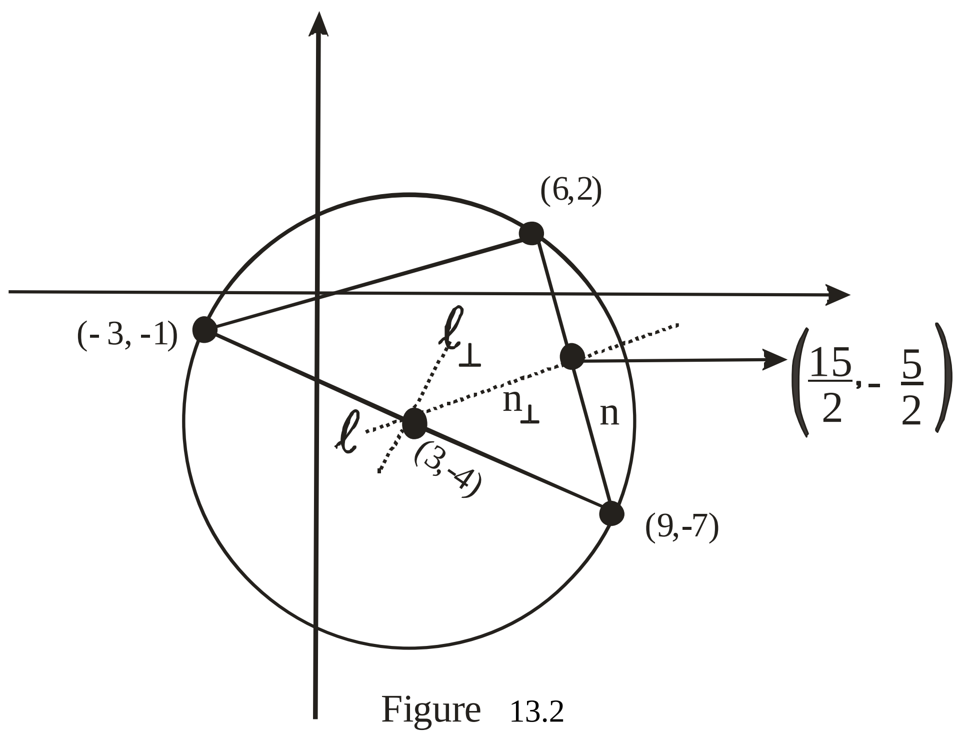


Figure 13.2

[Equation of line by point- slope foam is $y - y_1 = m(x - x_1)$]

Therefore the equation of $\ell_\perp$ and $n_\perp$ are

$\ell_{\perp}$: $y + 4 = 2(x - 3)$ or $2x - y = 10$

$n_{\perp}$: $y + \frac{5}{2} = \frac{1}{3}\left(x - \frac{15}{2}\right)$ or $x - 3y = 15$

Solving these two simultaneous equations we obtain x = 3, y = –4.

Hence the centre of the circle is (3, –4)

The radius is $r = \sqrt{(3+3)^2 + (-4+1)^2}$

$r = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$

Put the (h , k ) = (3 , -4 ) and $r = 3\sqrt{5}$ in equation $(x - h)^2 + (y - k)^2 = r^2$

We get , $(x - 3)^2 + (y + 4)^2 = (3\sqrt{5})^2$

Or $x^2 + y^2 - 6x + 8y - 20 = 0$

**Example 6: Find the equation of the circle which is tangent to the y – axis, which passes through the point (–1, –1), and the centre of which is on the line 2x + y + 4 = 0.**

**Solution:** If ( h, k) is the centre of the circle.

Since the circle is tangent to y – axis so that radius is

$r = h$

As circle passes through point (–1, –1), so

$(h + 1)^2 + (k + 1)^2 = h^2$

Or $k^2 + 2k + 2h + 2 = 0$ (1)

Since the centre (h, k) lines on the line 2x + y + 4 = 0

So $2h + k + 4 = 0$ (2)

from (2) $h = \frac{-(k+4)}{2}$ (3)

Put in (1)

$k^2 + 2k + 2\left[\frac{-(k+4)}{2}\right] + 2 = 0$

Or $k^2 + 2k - k - 4 + 2 = 0$

Or $k^2 + k - 2 = 0$

$k = 1$ or $k = -2$

From (3) when $k = 1$, $h = -\frac{5}{2}$

When $k = -2$, $h = -1$

Hence there are two circles with centres $\left(-\frac{5}{2}, 1\right)$, the radius $r = \sqrt{\left(-1+\frac{5}{2}\right)^2 + (-1-1)^2} = \frac{5}{2}$

and (–1, –2), the radius $r = \sqrt{(-1+1)^2 + (-1+2)^2} = 1$

The equation of circles are

$$\left(x+\frac{5}{2}\right)^2 + (y-1)^2 = \left(\frac{5}{2}\right)^2$$

Or $x^2 + y^2 + 5x - 2y + 1 = 0$ (4)

And, $(x+1)^2 + (y+2)^2 = 1$

Or $x^2 + y^2 + 2x + 4y + 4 = 0$ (5)

**Example7: Find the equation of the circle which contains the point (0, 1) and touches the line x + 2y + 2 = 0 at the point (4, –3).**

**Solution:** If the (h, k) is the centre of the circle. Since the points A (0, 1) and B (4, – 3) lie on the circle, so

$$|OA| = |OB|$$

$$\sqrt{(h-0)^2 + (k-1)^2} = \sqrt{(h-4)^2 + (k+3)^2}$$

Or $h^2 + h^2 - 2k + 1 = h^2 + k^2 - 8h + 6k + 25$

Or $8h - 8k - 24 = 0$

Or $h - k - 3 = 0$……………………….(1)

Slope of the line x + 2y + 2 = 0

Is $m = -\frac{1}{2}$

Slope of perpendicular OB is 2.

Equation of the perpendicular OB on the line is

$$\frac{k+3}{h-4} = 2$$

Or $2h - k - 11 = 0$ (2)

Solving equation (1) and (2) we have

$$h = 8, \quad k = 5$$

So the centre is (8, 5) and radius $r = \sqrt{(8-0)^2 + (5-1)^2} = \sqrt{80}$

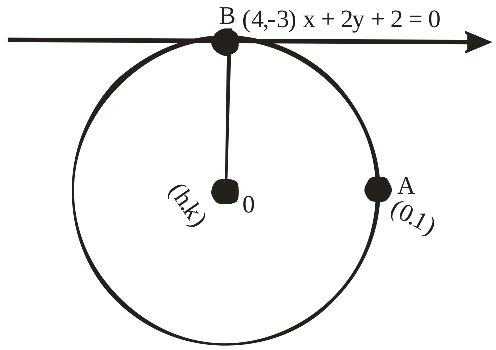


**Figure** 13.3

Hence the equation of the circle is

$$(x-8)^2 + (y-5)^2 = 80$$

Or $\quad x^2 + y^2 - 16x - 10y - 9 = 0$

**Example 8: A circle is tangent to the x – axis at (5, 0) and is also tangent to the line y = x. Find the centre, radius, and an equation of the circle.**

**Solution:** Since the circle is tangent to (5, 0), so the centre of the circle is (h, k) = (5, k).

The radius of the circle is r = k

The perpendicular distance of centre (5, k) from the line x – y = 0 is $k = \dfrac{|5-k|}{\sqrt{2}}$

Or, $\quad |5-k| = \sqrt{2}\,k$

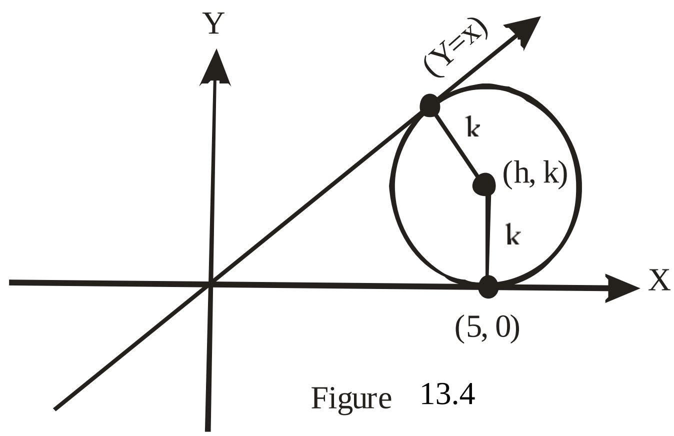


Figure 13.4

Squaring both sides,

$$k^2 - 10k + 25 = 2k^2$$

Or $$k^2 + 10k - 25 = 0$$

$$K = -5 + 2\sqrt{2} \quad \text{or} \quad 5 - 5\sqrt{2}$$

Hence the centre is at $\left(5,5\left(\sqrt{2}-1\right)\right)$

Radius is $r = 5\left(\sqrt{2}-1\right)$

An equation is $(x-5)^2 + \left[y - 5\left(\sqrt{2}-1\right)\right]^2 = \left(5\left(\sqrt{2}-1\right)\right)^2$

Or $x + y^2 - 10x - 10y\left(\sqrt{2}-1\right) + 25 = 0$ ………………….(1)

When the center is at $\left(5, -5\left(\sqrt{2}+1\right)\right)$

Radius is $r = 5\left(\sqrt{2}+1\right)$

An equation is $(x-5)^2 + \left(y + 5\left(\sqrt{2}+1\right)\right)^2 = \left(5\left(\sqrt{2}+1\right)\right)^2$

Or $x^2 + y^2 - 10x + 10y\left(\sqrt{2}+1\right) + 25 = 0$ ………………….(2)

## Exercise 13

Q.1: Find the equation of the circles with the given centres and radii.

**(a)** $(-1, 2)$, $r = \sqrt{2}$ **(b)** $(-\sqrt{2}, -2)$, $r = \sqrt{6}$

**(c)** $(0, 0)$, $r = a$ **(d)** $(1, -3)$, $r = 3$

Q.2: Find centres and radii of the circles with the following equations:

**(a)** $x^2 + y^2 - 6x + 6y = 0$ **(b)** $x^2 + y^2 - 4x + y - 1 = 0$

**(c)** $3x^2 + 3y^2 - 2x - 6y - 2 = 0$ **(d)** $(x + 2)^2 + (y - 1)^2 = 16$

(e) $x^2 + y^2 + -4x + 6y - 12 = 0$

Q.3: Find the equations of the circles:

(a) Passing through the points ( 1, 2), (0, –1) and (–1, 1).

(b) Passing through the points (0,1), ( 3, –3) and (3, –1).

(c) Through (–2,1), (–4, –3) and ( 3, 0).

(d) through (2, –1) and (–2,0) with center on $2x - y - 1 = 0$

(e) Through (–1, 2) and tangent to the axes.

(f) Through (3, 1) and touching the x – axis at (0,0).

Q.4: Find the equations of the following circles:

(a) through the point of intersection of the lines.

$2x - y + 7 = 0$ and $3x + y + 8 = 0$ with center at the origin

(b) Centre at the point of intersection of the lines.

$x - 2y + 4 = 0$ and $2x + y - 2 = 0$ with radius 4 units.

Q.5: Find the equations of the following circles:

(a) With center on the line $y = -x$ has radius 4 and passes through the origin.

(b) The circle touching the line $x = 2$ and $x = 12$ and passes through (4, 5).

(c) Through origin and whose intercepts on the axes are 3 and 4.

(d) through (–1, –2), ( 6, –1) and touching the x – axis.

Q.6: Find the equation of the following circles.

(a) Concentric with the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ with radius 6 units.

(b) Concentric with the circle $x^2 + y^2 - 7x + 8y = \frac{9}{2}$ and touches the y – axis.

Q.7: Find the equations of the following circles:

(a) Which is tangent to the positive x and y – axis and radius 5 units.

(b) Which touches both the axes of 4th quadrant and has a radius of 5 units.

(c) Whose center is ( –3, 2) and passes through the center of the circle $x^2 + y^2 – 4x + 8y – 16 = 0$

Q.8: Find which of the two circles $x^2 + y^2 – 3x + 4y = 0$ and $x^2 + y^2 – 6x – 8y =0$ is greater.

Q.9: Find the equation of the circle having:

(*a*) (–2, 5) and (3, 4) as the end points of its diameter. Find also its centre and radius.

(b) (–3, 7) and (2,- 1) as the end points of its diameter. Find also its centre and radius.

Q.10: Find the equation of the circle whose center is at (–2, 5) and which touches the line $4x – 3y – 18 = 0$

Q.11: Show that the circles $x^2 + y^2 +2x – 2y – 7 = 0$ and $x^2 + y^2 – 6x + 4y + 9 = 0$ touch externally.

Q.12: Show that the circles $x^2 + y^2 +2x – 8 = 0$ and $x^2 + y^2 – 6x + 6y – 46 = 0$ touch internally.

## Answers

**Q.1:** (a) $x^2 + y^2 + 2x – 4y + 3 =0$

(b) $x^2 + y^2 + 2\sqrt{2}x + 4y = 0$ (c) $x^2 + y^2 = a^2$

(d) $(x – 1)^2 + (y + 3)^2 = 16$

**Q.2:** (a) $(3, – 3), r = 3\sqrt{2}$ (b) $\left(2, -\frac{1}{2}\right), r = \frac{1}{2}\sqrt{21}$

(c) $\left(\frac{1}{3}, 1\right), r = 4/3$ (d) (- 2 , 1) , r = 4 (e) (2 , -3) , r =5

**Q.3**: (a) $x^2 + y^2 – x – y – 2 =0$

(b) $3x^2 + 3y^2 – x + 12y – 15 = 0$

(c) $48x^2 + 48y^2 – 784x + 632y – 2440 = 0$

(d) $\left(x+\frac{1}{4}\right)^2 + \left(y+\frac{3}{2}\right)^2 = \frac{85}{16}$

(e) $(x+1)^2 + (y-1)^2 = 1$ and $(x+5)^2 + (y-5)^2 = 25$

(f) $x^2 + y^2 - 10y = 0$

**Q.4:** (a) $x^2 + y^2 = 10$ (b) $x^2 + y^2 - 4y - 12 = 0$

**Q.5:** (a) $(x + 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 = 16$

(b) $(x-7)^2 + (y-4)^2 = 25$ and $(x-7)^2 + (y+4)^2 = 25$

(c) $\left(x-\frac{3}{2}\right)^2 + (y-2)^2 = \frac{25}{4}$

(d) $(x-3)^2 + (y+5)^2 = (5)^2$ and $(x-23)^2 + (y+145)^2 = (145)^2$

**Q.6:** (a) $x^2 + y^2 - 6x + 4y - 23 = 0$

(b) $x^2 + y^2 - 7x + 8y + 16 = 0$

**Q.7:** (a) $(x-5)^2 + (y-5)^2 = (5)^2$

(b) $(x-5)^2 + (y+5)^2 = (5)^2$

(c) $x^2 + y^2 + 6x - 4y - 48 = 0$

**Q.8:** **Second circle to greater than first.**

**Q.9:** (a) $x^2 + y^2 - x - 9y + 14 = 0$ (b) $4x^2 + 4y^2 + 4x - 24y - 141 = 0$

**Q.10:** $x^2 + y^2 + 6x - 10y - 47 = 0$

## Short Questions

**Write the short answers of the following**

Q.1: Write the equation of circle with centre at (h, k) and radius r.

Q.2: Write the general form of the circle, also represent the centre and radius is this form.

Q.3: Find the equation of circle with centre (0, 0) and radius r.

Q.4: Find the equation of circle with centre (-3, 4) and radius 4.

Q.5: Find the equation of circle with centre on origin and radius is $\frac{1}{2}$.

Q.6: Find centre and radius of the circle $x^2 + y^2 + 9x - 7y - 33 = 0$

Q.7: Find the centre and radius of the circle $6x^2 + 6y^2 - 18y = 0$

Q.8: What type of circle is represented by $x^2 + y^2 - 2x + 4y + 8 = 0$

Q.9: Find the equation of circle with centre at (-1, 3) and tangent to x –axis.

Q.10: Find the equation of circle with centre (3, 0) and tangent to y – axis.

Q.11: Find the equation of the circle touches the lines at x = 0 and x = 10 and the centre is an x – axis.

Q.12: Reduce the equation into standard form $x^2 + y^2 - 4x + 6y - 12 = 0$

Q.13: Reduce the equation into standard form $2x^2 + 2y^2 - 5x + 4y - 7 = 0$

Q.14: Reduce the equation into standard form $x^2 + y^2 - 10y = 0$.

Q.15: Find the equation of circle centered at (-3, 2) and passes through the point (2, -4).

Q.16: Define the circle.

## Answers

Q2. Centre $(-g, -f)$, $r = \sqrt{g^2 + f^2 - c}$

Q3. Equation of circle is: $x^2 + y^2 = r^2$

Q4. $x^2 + y^2 + 6x - 8y + 9 = 0$

Q5. $x^2 + y^2 - \frac{1}{4} = 0$

Q6. centre $= \left(-\frac{7}{2}, \frac{7}{2}\right)$, $r = \sqrt{\frac{131}{2}}$

Q7. Centre $= \left(0, \frac{3}{2}\right)$, $r = \frac{3}{2}$

Q8. $r^2 < 0$ imaginary circle.

Q9. $x^2 + y^2 + 2x - 6y + 1 = 0$

Q10. $x^2 + y^2 - 6x = 0$

Q11. $x^2 + y^2 - 10x = 0$

Q12. $(x - 2)^2 + (y + 3)^2 = 5^2$

Q13. $\left(x - \frac{5}{4}\right)^2 + (y + 1)^2 = \left(\frac{\sqrt{41}}{8}\right)^2$

Q14. $(x - 0)^2 + (y - 5)^2 = 5^2$

Q15. $x^2 + y^2 + 6x - 4y - 48 = 0$

## Objective Type Questions

**Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.**

__**1. General Equation of the circle is:**

(a) $x^2 + y^2 + 2gx + 2fy + c = 0$ (b) $(x - h)^2 + (y - k)^2 = r^2$

(c) $x^2 + y^2 + x + y + 1 = 0$ (d) None of these

__**2. Standard equation of the circle is:**

(a) $x^2 + y^2 + 2gh + 2fy + c = 0$ (b) $(x - h)^2 + (y - k)^2 = r^2$

(c) $x^2 + y^2 + x + y + 1 = 0$ (d) None of these

__**3. Straight line from center to the circumference is:**

(a) Circle (b) Radius

(c) Diameter (d) None of these

__**4. Radius of the circle $x^2 + y^2 = 1$ is:**

(a) 1 (b) (0, 0)

(c) 2 (d) None of these

__**5. Radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is:**

(a) c (b) $c^2$

(c) $\sqrt{g^2 + f^2 - c}$ (d) None of these

__**6. Centre of the circle $x^2 + y^2 - 2x - 4y = 8$ is:**

(a) (1, 2) (b) (2, 4)

(c) (1, 3) (d) None of these

__**7. Radius of the circle $x^2 + y^2 - 2x - 4y = 8$ is:**

(a) 8 (b) $\sqrt{8}$

(c) $\sqrt{12}$ (d) None of these

__**8. Equation of the unit circle is:**

(a) $x^2 + y^2 + 2x + 2y + 1 = 0$ (b) $x^2 + y^2 = 1$

(c) $x^2 + y^2 = r^2$ (d) None of these

__**9. Radius of the circle $(x - 1)^2 + (y - 2)^2 = 16$ is:**

(a) 2 (b) 1

(c) 4 (d) None of these

**__10. Centre of the circle $(x - 1)^2 + (y - 2)^2 = 16$ is :**

(a) (1, 2) (b) (2, 1)

(c) (4, 0) (d) None of these

## Answers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** | a | **2.** | b | **3.** | b | **4.** | a | **5.** | c |
| **6.** | a | **7.** | d | **8.** | b | **9.** | d | **10.** | a |

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