Chapter 4
General Identities

4.1 Introduction:

In the previous chapters, we have dealt with functions of one angle. In this chapter we will discuss the trigonometrical ratios of the sum and difference of any two angles in terms of the ratios of these angles themselves. We will also derives several formulas for this purpose and point out some of their more elementary uses.

4.2 Distance formula:

Let P(x_1 , y_1) and Q(x_2 , y_2) be two points . If "d" denotes the distance between them, then

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., sum of the square of the difference of x-coordinates and y- coordinates and then the square roots.

Example1: Find distance between the points P(5 , 7) and Q(-3 , 4)

Solution:

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3 - 5)^2 + (4 - 7)^2}$$

$$= \sqrt{(-2)^2 + (-3)^2}$$

$$= \sqrt{4+9} \quad = \quad \sqrt{13}$$

4.3 Fundamental law of trigonometry

Let α and β be any two angles (real numbers), then

$$\text{Cos}(\alpha - \beta) = \cos\alpha \cos\beta \;-\; \sin\alpha \sin\beta$$

Which is called the **Fundamental law of trigonometry**

Proof: for convenience, let us assume that $\alpha > \beta > 0$

Consider a unit circle with centre at origin O.

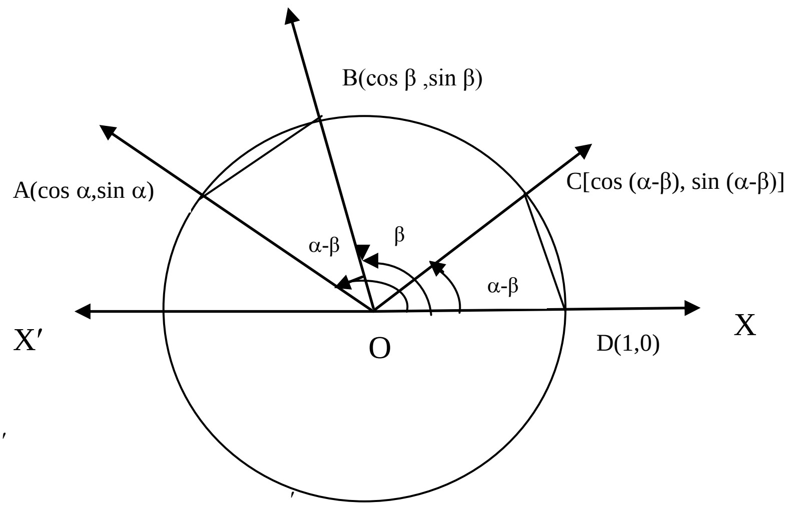
Let the terminal side of angles α and β

cut the unit circle at A and B respectively.

Evidently $\angle AOB = \alpha - \beta$. Take a point C

on the unit circle so that $\angle XOC = \angle AOB = \alpha - \beta$

Join A , B and C , D.



Now angle α , β and $\alpha - \beta$ are in standard position.

$\therefore$ The coordinates of A are ($\cos\alpha$, $\sin\alpha$)

The coordinates of B are ($\cos\beta$, $\sin\beta$)

The coordinates of C are $[\cos(\alpha-\beta) , \sin(\alpha - \beta)]$

and the coordinates D are (1 , 0)

Now Δ AOB and ΔCOD are congruent.

$\therefore \quad |AB| = |CD|$

$\Rightarrow \quad |AB|^2 = |CD|^2$

Using the distance formula , we have:

$$(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = [\cos (\alpha- \beta) - 1]^2 + [\sin (\alpha - \beta)- 0]^2$$

$$\Rightarrow \cos^2 \alpha + \cos^2\beta - 2\cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta - 2\sin \alpha \sin \beta$$

$$= \cos^2 (\alpha - \beta) +1 - 2 \cos (\alpha - \beta) + \sin^2 (\alpha - \beta)$$

$$\Rightarrow 2 - 2 (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = 2 - 2 \cos(\alpha - \beta)$$

Hence,

$$\boxed{\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \ldots\ldots\ldots\ldots\ldots(1)}$$

Note: Although we have proved this law for $\alpha > \beta > 0$, it is true for all values of α and β

Suppose we know the values of sin and cos of two angles α and β, we can find

$\cos(\alpha - \beta)$ using this law as explained in the following example:

Example 1:

Find the value of Cos 15^o.

Solution:

$$\text{Cos } 15^o = \text{Cos } (45^o - 30^o)$$
$$= \text{Cos } 45^o \text{ Cos } 30^o + \text{Sin } 45^o \text{ Sin } 30^o$$
$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

4.4 Deductions from fundamental law:

(i) Prove that : $\text{Cos}(-\beta) = \text{Cos}\,\beta$

Put $\alpha = 0$ in above equation (1) , then

$$\text{Cos } (0 - \beta) = \text{Cos } 0 \text{ Cos } \beta + \text{Sin } 0 \text{ Sin}\,\beta$$
$$\text{Cos}(-\beta) = 1.\text{ Cos } \beta + 0.\text{ Sin}\,\beta \qquad \because \text{ Cos } 0 = 1$$
$$\text{Cos}(-\beta) = \text{Cos}\,\beta \qquad \because \text{ Sin } 0 = 0$$

(ii) Prove that : $\text{Cos}\left(\frac{\pi}{2} - \beta\right) = \text{Sin}\,\beta$

Putting $\alpha = \pi/2$ in equation

$$\text{Cos}(\alpha - \beta) = \text{Cos}\alpha\ \text{Cos}\beta + \text{Sin}\alpha\ \text{Sin}\beta$$
$$\text{Cos}\left(\frac{\pi}{2} - \beta\right) = \text{Cos}\frac{\pi}{2}\ \text{Cos}\beta + \text{Sin}\frac{\pi}{2}\ \text{Sin}\beta$$
$$= 0\ .\ \text{Cos}\beta + 1\ .\ \text{Sin}\beta$$
$$\boxed{\text{Cos}\left(\frac{\pi}{2} - \beta\right) = \text{Sin}\,\beta}$$

(iii) Prove that : $\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$

Put $\beta = -\frac{\pi}{2}$ in equation

$$\text{Cos}(\alpha-\beta) = \text{Cos}\alpha\ \text{Cos}\beta - \text{Sin}\alpha\ \text{Sin}\beta$$

$$\text{Cos}[(\alpha - (-\frac{\pi}{2})] = \cos\alpha \,.\, \cos(-\frac{\pi}{2}) + \sin\alpha \,.\, \sin(-\frac{\pi}{2})$$

$$\cos(\alpha + \frac{\pi}{2}) = \cos\alpha \,.\, 0 + \sin\alpha \,.\, (-1)$$

$$\boxed{\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha}$$

(iv) Prove that: $\sin(-\beta) = -\sin\beta$

By (iii) we have $\cos(\frac{\pi}{2} + \beta) = -\sin\beta$

replace β by $-\beta$

$$\cos(\frac{\pi}{2} - \beta) = -\sin(-\beta)$$

$$\sin\beta = -\sin(-\beta) \qquad [\text{ by (ii)}]$$

$$\boxed{\mathbf{\sin(-\beta) = -\sin\beta}}$$

(v) Prove that : $\sin(\frac{\pi}{2} + \alpha) = \cos\alpha$

we know that $\text{Cos}\left(\frac{\pi}{2} - \beta\right) = \text{Sin}\ \beta$

putting $\beta = \frac{\pi}{2} + \alpha$ in above equation, we get

$$\text{Cos}[\frac{\pi}{2} - (\frac{\pi}{2} + \alpha)] = \sin(\frac{\pi}{2} + \alpha)$$

$\Rightarrow \quad \cos(-\alpha) = \sin\left(\frac{\pi}{2} + \alpha\right)$

$\cos\alpha = \sin\left(\frac{\pi}{2} + \alpha\right)$

$$\boxed{\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha}$$

(vi) Prove that : $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Since $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

replacing β by $-\beta$, we get

$\cos[\alpha-(-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$

{because $\cos(-\beta) = \cos\beta$,

$\sin(-\beta) = \sin\beta$}

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

(vii) Prove that : $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

We know that

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

replace α by $\frac{\pi}{2} + \alpha$, we get

$\cos[(\frac{\pi}{2} + \alpha) + \beta] = \cos(\frac{\pi}{2} + \alpha)\cos\beta - \sin(\frac{\pi}{2} + \alpha)\sin\beta$

$\cos[\frac{\pi}{2} + (\alpha + \beta)] = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$

$-\sin(\alpha + \beta) = -[\sin\alpha\cos\beta + \cos\alpha\sin\beta]$

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

(viii) Prove that: $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

We know that

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

replacing β by $-\beta$, we get

$$\sin(\alpha - \beta) = \sin\alpha \cos(-\beta) + \cos\alpha \sin(-\beta)$$

{because $\cos(-\beta) = \cos\beta$,

$\sin(-\beta) = \sin\beta$}

$$\boxed{\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta}$$

(ix) Prove that: $\text{Sin}\left(\dfrac{\pi}{2} + \beta\right) = -\text{Cos}\ \beta$

Put $\alpha = \dfrac{\pi}{2}$ in

$$\text{Sin}(\alpha + \beta) = \text{Sin}\alpha\ \text{Cos}\beta + \text{Cos}\alpha\ \text{Sin}\beta$$

$$\text{Sin}\left(\frac{\pi}{2} + \beta\right) = \text{Sin}\frac{\pi}{2}\ \text{Cos}\beta + \text{Cos}\frac{\pi}{2}\ \text{Sin}\beta$$

$$= 0\ .\ \text{Cos}\beta + 0\ .\ \text{Sin}\beta$$

$$\boxed{\text{Sin}\left(\frac{\pi}{2} + \beta\right) = \text{Cos}\ \beta}$$

(x) Prove that: $\text{Sin}\left(\dfrac{\pi}{2} - \beta\right) = \text{Cos}\beta$

Put $\alpha = \dfrac{\pi}{2}$ in

$$\text{Sin}(\alpha - \beta) = \text{Sin}\alpha\ \text{Cos}\beta - \text{Cos}\alpha\ \text{Sin}\beta$$

$$\text{Sin}\left(\frac{\pi}{2}-\beta\right)= \text{Sin}\frac{\pi}{2}\ \text{Cos}\beta-\text{Cos}\frac{\pi}{2}\text{Sin}\beta$$

$$\text{Sin}\left(\frac{\pi}{2}-\beta\right)= 1.\text{Cos}\beta-0.\text{Sin}\beta$$

$$\boxed{\text{Sin}\left(\frac{\pi}{2}-\beta\right)= \text{Cos}\beta}$$

(xi) $\tan(\alpha+\beta) = \dfrac{\text{Sin}\,(\alpha+\beta)}{\text{Cos}(\alpha+\beta)}$

$$=\frac{\text{Sin}\ \alpha\ \text{Cos}\beta+\text{Cos}\ \alpha\ \text{Sin}\ \beta}{\text{Cos}\alpha\ \text{Cos}\beta-\text{Sin}\ \alpha\ \text{Sin}\ \beta}$$

Divide numerator and denominator by $\text{Cos}\alpha\ \text{Cos}\beta$, we get

$$\tan(\alpha+\beta) = \frac{\dfrac{\text{Sin}\ \alpha\ \text{Cos}\beta}{\text{Cos}\alpha\ \text{Cos}\beta}+\dfrac{\text{Cos}\ \alpha\ \text{Sin}\ \beta}{\text{Cos}\alpha\ \text{Cos}\beta}}{1-\dfrac{\text{Sin}\ \alpha\ \text{Cos}\beta}{\text{Cos}\alpha\ \text{Cos}\beta}}$$

$$\boxed{\mathbf{tan}(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

(xii) $\text{Cot}\,(\alpha+\beta) = \dfrac{1}{\tan(\alpha+\beta)}$

$$=\frac{1-\tan\alpha\ \tan\beta}{\tan\alpha+\tan\beta}$$

$$=\frac{1-\dfrac{1}{\text{Cot}\alpha\ \text{Cot}\beta}}{\dfrac{1}{\text{Cot}\alpha}+\dfrac{1}{\text{Cot}\beta}}$$

$$\boxed{\textbf{Cot}(\boldsymbol{\alpha}+\boldsymbol{\beta})=\frac{\text{Cot}\alpha\ \text{Cot}\beta-1}{\text{Cot}\alpha+\text{Cot}\beta}}$$

Similarly, $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

and $\cot(\alpha-\beta)=\dfrac{\cot\alpha\ \cot\beta+1}{\cot\alpha-\cot\beta}$

($\tan\theta$ and Cot θ are odd functions).

Note :

(1) If θ is added or subtracted from odd multiple of right angle ($\pi/2$), the trigonometric ratios change into co-ratios and vice-versa.

e.g.,

Sin $(\frac{\pi}{2}-\theta)=\cos\theta$, cos $(\frac{\pi}{2}-\theta)=\sin\theta$, tan $(\frac{\pi}{2}-\theta)=\cot\theta$

Sin$(\frac{3\pi}{2}-\theta)=-\cos\theta$, cos$(\frac{3\pi}{2}-\theta)=-\sin\theta$, tan$(\frac{3\pi}{2}-\theta)=\cot\theta$

(2) If θ is added or subtracted from an even multiple of right angle ($\pi/2$), the trigonometric ratios shall remain the same.

e.g.,

Sin $(\pi-\theta)=\sin\theta$, cos $(\pi-\theta)=-\cos\theta$, tan $(\pi-\theta)=-\tan\theta$

Sin$(2\pi-\theta)=-\sin\theta$, cos$(2\pi-\theta)=\cos\theta$, tan$(2\pi-\theta)=-\tan\theta$

Example 2:

Show that Cos $(180^o+\theta)=-$Cos θ

Solution:

$$\begin{aligned}\text{L.H.S} &= \text{Cos}\ (180^o+\theta)\\ &= \text{Cos}\ 180^o\ \text{Cos}\,\theta-\text{Sin}\ 180^o\ \text{Sin}\ \theta\\ &= (-1)\ \text{Cos}\,\theta-(0)\ \text{Sin}\,\theta\\ &= \text{Cos}\,\theta-0\\ &= -\text{Cos}\,\theta = \text{R.H.S}\end{aligned}$$

Example 3:

If $\text{Sin}\alpha=\dfrac{4}{5}$ and $\text{Sin}\beta=\dfrac{12}{13}$, neither terminal ray of α nor β is in the first quadrant, find Sin$(\alpha+\beta)$.

Solution:

Because $\text{Cos}\alpha = \sqrt{1-\text{Sin}^2\alpha}$

$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$$

$$\text{Cos}\alpha = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Since α and β does not lie in 1st quadrant and Sin α and Sin β is positive, therefore α, β lies is 2nd quadrant and in 2nd quadrant Cos α is negative is

$$\text{Cos}\alpha = -\frac{3}{5}$$

Then $\text{Cos}\beta = \sqrt{1-\text{Sin}^2\beta}$

$$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}}$$

$$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

$$\therefore\ \text{Cos}\beta = -\frac{5}{13} \qquad \because \beta \text{ lies in } 2^{nd} \text{ quadrant}$$

Now $\text{Sin}(\alpha+\beta) = \text{Sin}\alpha\ \text{Cos}\beta + \text{Cos}\alpha\ \text{Sin}\beta$

$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \frac{36}{65} = \frac{-20-36}{65} = -\frac{56}{65}$$

$$\text{Sin}(\alpha+\beta) = -\frac{56}{65}$$

Example 4:

Express $4\,\text{Sin}\ \theta + 7\,\text{Cos}\ \theta$ in the form r Sin $(\theta + \phi)$, where the terminal side of θ and ϕ are in the first quadrant.

Solution: Multiplying and Dividing the expression by

$$r = \sqrt{(4)^2+(7)^2} = \sqrt{16+49} = \sqrt{65}$$

$$4\ \text{Sin}\ \theta + 7\text{Cos}\ \theta = \sqrt{65}\left[\frac{4}{\sqrt{65}}\text{Sin}\ \theta + \frac{7}{\sqrt{65}}\text{Cos}\ \theta\right]$$

$$= \sqrt{65}\left[\text{Sin }\theta\left(\frac{4}{\sqrt{65}}\right) + \text{Cos }\theta\left(\frac{7}{\sqrt{65}}\right)\right] \quad ..\ (1)$$

Since $\quad r\ \text{Sin}(\theta + \phi) = r(\text{Sin }\theta\ \text{Cos }\phi + \text{Cos }\theta\ \text{Sin }\phi) \quad ..\ (2)$

Where $\quad 0 < \phi < \dfrac{\pi}{2}$

Let $\quad \text{Cos}\phi = \dfrac{4}{\sqrt{65}}$ and $\text{Sin}\phi = \dfrac{4}{\sqrt{65}}$, then

$$4\ \text{Sin }\theta + 7\ \text{Cos }\theta = \sqrt{65}\left[\text{Sin}\theta\ \text{Cos}\phi + \text{Cos}\theta\ \text{Sin}\phi\right]$$

$$= \sqrt{65}\left[\text{Sin }(\theta + \phi)\right]$$

Where $\text{Cos}\phi = \dfrac{4}{\sqrt{65}}$ and $\text{Sin}\phi = \dfrac{4}{\sqrt{65}}$

i.e., $\quad \tan\phi = \dfrac{7}{4}$

$$\phi = \tan^{-1}\frac{7}{4}$$

Example 5:

Find the value of Sin 75^o.

Solution: $\text{Sin}(75^o) = \text{Sin }(45^o + 30^o)$

$$= \text{Sin }45^o\ \text{Cos }30^o + \text{Cos }45^o\ \text{Sin }30^o$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 6:

$$\text{Sin }(\alpha+\beta).\ \text{Sin}(\alpha-\beta) = \text{Cos}^2\beta - \text{Cos}^2\alpha$$

Solution: L.H.S $= \text{Sin }(\alpha+\beta).\ \text{Sin}(\alpha-\beta)$

$$= \left[\text{Sin}\alpha\ \text{Cos}\beta + \text{Cos}\alpha\ \text{Sin}\beta\right]$$

$$\left[\text{Sin}\alpha\ \text{Cos}\beta - \text{Cos}\alpha\ \text{Sin}\beta\right]$$

$$= \text{Sin}^2\alpha\ \text{Cos}^2\beta - \text{Cos}^2\alpha\ \text{Sin}^2\beta$$

$$= (1 - \cos^2\alpha)\cos^2\beta - \cos^2\alpha\,(1 - \cos^2\beta)$$

$$= \text{Cos}^2\beta - \text{Cos}^2\alpha\ \text{Cos}^2\beta - \text{Cos}^2\alpha + \text{Cos}^2\alpha\ \text{Cos}^2\beta$$

$$= \text{Cos}^2\beta - \text{Cos}^2\alpha$$

= R.H.S

Exercise 4.1

Q.1 Find the value of (i) $\cos 75^0$ (ii) $\sin 15^0$ (iii) $\sin 105^0$ (iv) $\cos 105^0$ (v) $\tan 105^0$.

Q.2 Prove that:

(i) $\sin(180^0 - \theta) = \sin\theta$ (ii) $\cos(270^0 + \theta) = \sin\theta$

(iii) $\tan(180^0 + \theta) = \tan\theta$ (iv) $\sin(360^0 - \theta) = -\sin\theta$

(v) $\cot(360^0 + \theta) = \cot\theta$ ((vi) $\tan(90^0 + \theta) = -\cot\theta$

Q.3 Show that:

(i) $\sin(x - y)\cos y + \cos(x - y)\sin y = \sin x$

(ii) $\cos(x + y)\cos y + \sin(x + y)\sin y = \cos x$

(iii) $\cos(A + B)\sin(A - B) = \sin A\cos A - \sin B\cos B$

(iv) $\dfrac{\tan(x+y) - \tan x}{1 + \tan(x+y)\tan x} = \dfrac{\sin x}{\cos y}$

Q.4 Suppose that A , B and C are the measure of the angles of a triangle such that $A + B + C = \pi$, prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Q.5 Prove that:

(i) $\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

(ii) $\sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + 30^0)$

(iii) $\tan(45^0 - \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

(iv) $\tan(45^0 + \theta) = \dfrac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}$

(v) $\dfrac{\tan(\alpha + \beta)}{\cot(\alpha - \beta)} = \dfrac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha\,\tan^2\beta}$

(vi) $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

(vii) $\dfrac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

(viii) $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

(ix) $\tan(x + \frac{\pi}{4}) - \tan\left(x - \frac{3\pi}{4}\right) = 0$

(x) $\text{Cos}(\frac{\pi}{3} + x) - \sin\left(\frac{\pi}{6} - x\right) = 0$

Q.6 Prove that:

(i) $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$

(ii) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

Q.7 If $\text{Sin}\,\alpha = \frac{4}{5}$ and $\text{Sin}\,\beta = \frac{12}{13}$, both α and β are in the 1st quadrant find:

(i) $\sin(\alpha - \beta)$ (ii) $\cos(\alpha + \beta)$

Q.8 If $\cos A = \frac{1}{5}$ and $\cos B = \frac{1}{2}$ A and B be acute angles , find the value of: (i) $\sin(A + B)$ (ii) $\cos(A - B)$

Q.9 If $\tan\alpha = \frac{3}{4}$ and $\sec\beta = \frac{13}{5}$ and neither α nor β is in the 1st quaderant , find $\sin(\alpha + \beta)$

Q.10 Prove that : $\dfrac{\sin\alpha}{\sec 4\alpha} + \dfrac{\cos\alpha}{\operatorname{cosec} 4\alpha} = \sin 5\alpha$

Q.11 If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$

Q.12 If α , β and γ are the angle of triangle ABC , then prove that

(i) $\sin(\alpha + \beta) = \sin\gamma$ (ii) $\cos(\alpha + \beta) = -\sin\gamma$

(iii) $\tan(\alpha + \beta) + \tan\gamma = 0$

Q.13 (i)If $\cos\alpha = \frac{1}{7}$, $\cos\beta = \frac{13}{14}$, then prove that $\alpha - \beta = 60^0$,

where the terminal rays of α and β are in 1st quadrants.

(ii)If If $\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$, then prove that $\alpha + \beta = 45^0$,

where the terminal rays of α and β are in 1st quadrants.

Q.14 Express the following in the form of $r\sin(\theta + \phi)$, where the terminal rays of θ is in the 1st quadrants. Be sure to specify ϕ :

(i) $4\sin\theta + 3\cos\theta$ (ii) $\sqrt{3}\sin\theta + \sqrt{7}\cos\theta$

(iii) $5\sin\theta - 4\cos\theta$ (iv) $\sin\theta + \cos\theta$

Answers 4.1

Q.1 (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (iii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(iv) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (v) $\frac{\sqrt{3}+1}{1-\sqrt{3}}$

Q.7 (i) $-\frac{16}{25}$ (ii) $-\frac{33}{65}$

Q.8 (i) $\frac{\sqrt{24}+\sqrt{3}}{10}$ (ii) $\frac{1+6\sqrt{2}}{10}$

Q.9 $\frac{33}{65}$

Q.14 (i) $5\sin(\theta + \phi)$, $\phi = \tan^{-1}\left(\frac{3}{4}\right)$

(ii)$10\sin(\theta + \phi)$, $\phi = \tan^{-1}\left(\sqrt{\frac{7}{3}}\right)$

(iii) $\sqrt{41}\sin(\theta+\phi)$, $\phi = \tan^{-1}\left(\frac{4}{5}\right)$

(iv) $\sqrt{2}\sin(\theta+\phi)$, $\phi = \tan^{-1}(1)$

4.5 Double Angle Identities:

We know that:

$$\text{Sin}(\alpha+\beta) = \text{Sin}\alpha\ \text{Cos}\ \beta + \text{Cos}\alpha\ \text{Sin}\beta$$

Putting $\beta = \alpha$, we have

$$\boxed{\text{Sin}\ 2\alpha = 2\ \text{Sin}\alpha\ \text{Cos}\alpha}$$

$$\text{Sin}(\alpha+\alpha) = \text{Sin}\alpha\ \text{Cos}\alpha + \text{Cos}\alpha\ \text{Sin}\alpha$$

Also $\text{Cos}(\alpha+\alpha) = \text{Cos}\alpha\ \text{Cos}\beta - \text{Sin}\alpha\ \text{Sin}\beta$

Again putting $\beta = \alpha$ in this formula, we have

$$\text{Cos}(\alpha+\alpha) = \text{Cos}\alpha\ \text{Cos}\alpha - \text{Sin}\alpha\ \text{Sin}\alpha$$

$$\boxed{\text{Cos}2\alpha = \text{Cos}^2\alpha - \text{Sin}^2\alpha}$$

$$= \text{Cos}^2\alpha - (1-\text{Cos}^2\alpha) = \text{Cos}^2\alpha - 1 + \text{Cos}^2\alpha$$

$$\boxed{\text{Cos}\ 2\alpha = 2\text{Cos}^2\alpha - 1}$$

Again $\text{Cos}\ 2\alpha = \text{Cos}^2\alpha - \text{Sin}^2\alpha$

$$= 1 - \text{Sin}^2\alpha - \text{Sin}^2\alpha$$

$$\boxed{\text{Cos}\ 2\alpha = 1 - 2\ \text{Sin}^2\alpha}$$

Now, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

Putting $\beta = \alpha$ in this formula, we have

$$\tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1-\tan\alpha\tan\alpha}$$

$$\boxed{\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}}$$

4.6 Half Angle identities:

We know that:

$$\text{Cos}\ 2\alpha = 1 - 2\ \text{Sin}^2\alpha$$

Therefore $$2\,\text{Sin}^2\alpha = 1-\text{Cos}\,2\alpha$$

$$\text{Sin}^2\alpha = \frac{1-\text{Cos}\,2\alpha}{2}$$

$$\text{Sin}\,\alpha = \pm\sqrt{\frac{1-\text{Cos}\,2\alpha}{2}}$$

Putting $2\alpha = \theta \Rightarrow \alpha = \dfrac{\theta}{2}$ in this formula

We have $$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos 2(\frac{\theta}{2})}{2}}$$

$$\boxed{\text{Sin}\,\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \text{..........(i)}}$$

Also we know that

$$\text{Cos}\,2\alpha = 2\,\text{Cos}^2\alpha - 1$$

$\therefore$ $$\text{Cos}^2\alpha = \frac{1+\text{Cos}\,2\alpha}{2}$$

Or $$\text{Cos}\,\alpha = \pm\sqrt{\frac{1+\text{Cos}\,2\alpha}{2}}$$

Put $2\alpha = \theta \Rightarrow \alpha = \dfrac{\theta}{2}$ in this formula we have

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos 2(\frac{\theta}{2})}{2}}$$

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta)}{2}} \quad \text{.........(ii)}}$$

Now , $$\tan\frac{\theta}{2} = \frac{\text{Sin}\,\frac{\theta}{2}}{\text{Cos}\,\frac{\theta}{2}}$$

From (i) and (ii) $$= \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\boxed{\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\text{Cos }\theta}{1+\text{Cos }\theta}}}$$

Example 1:

If $\text{Sin }\theta = \frac{4}{5}$ and the terminal ray of θ is in the second quadrant. Find the value of (i) $\text{Sin } 2\theta$ (ii) $\text{Cos }\frac{\theta}{2}$

Solution:

Because $$\text{Cos }\theta = \sqrt{1-\text{Sin}^2\theta}$$
$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$$
$$= \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$\text{Cos }\theta = -\frac{3}{5}$ because the terminal ray of θ is in 2nd quadrant.

(i) $$\text{Sin } 2\theta = 2\text{ Sin }\theta\text{ Cos }\theta$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

(ii) $$\text{Cos }\frac{\theta}{2} = \sqrt{\frac{1+\text{Cos }\theta}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{2}{5}}{2}}$$
$$= \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}}$$

4.7 Triple angle identities:

(i) $\text{Cos } 3\theta = 4\text{ Cos}^3\theta - 3\text{ Cos }\theta$

(ii) $\text{Sin } 3\theta = 3\sin\theta - 4\sin^3\theta$

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Prove that :

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\begin{aligned}
\text{L.H.S.} &= \cos 3\theta \\
&= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \quad = \text{R.H.S.}
\end{aligned}$$

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\begin{aligned}
\text{L.H.S.} &= \sin 3\theta \\
&= \sin(2\theta + \theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta \quad = \text{R.H.S.}
\end{aligned}$$

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$$\begin{aligned}
\text{L.H.S.} &= \tan 3\theta \\
&= \tan(2\theta + \theta)
\end{aligned}$$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} \qquad = \frac{\dfrac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1 - \tan^2\theta}\tan\theta}$$

$$\left(\because \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}\right)$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad = \text{R.H.S.}$$

Example 2:

Show that $\text{Cosec } 2\theta - \text{Cot } 2\theta = \tan\theta$

Solution:

$$\begin{aligned}
\text{L.H.S} &= \text{Cosec } 2\theta - \text{Cot } 2\theta \\
&= \frac{1}{\text{Sin } 2\theta} - \frac{\text{Cos } 2\theta}{\text{Sin } 2\theta} \\
&= \frac{1-\text{Cos } 2\theta}{\text{Sin } 2\theta} \\
&= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} \qquad \because \ \text{Cos } 2\theta = 1-2\ \text{Sin}^2\theta \\
&= \frac{2\ \text{Sin}^2\theta}{2\ \text{Sin}\theta\ \text{Cos}\theta} = \frac{\text{Sin}\theta}{\text{Cos}\theta} \qquad \because \ \text{Sin } 2\theta = 2\ \text{Sin}\theta\ \text{Cos}\theta \\
&= \tan\theta = \text{R.H.S}
\end{aligned}$$

Example 3:

Using Half angle formula find

(i)Sin 210^o (ii) Cos 210^o (iii) tan 210^0

Solution:

(i) $$\text{Sin}\frac{\theta}{2} = \pm\sqrt{\frac{1-\text{Cos}\theta}{2}}$$

$$\text{Sin } 210^o = \pm\sqrt{\frac{1-\text{Cos } 420^o}{2}} = \pm\sqrt{\frac{1-\text{Cos } 60^o}{2}}$$

$$= \pm\sqrt{\frac{1-\frac{1}{2}}{2}} = \pm\sqrt{\frac{\frac{1}{2}}{2}} = \pm\sqrt{\frac{1}{4}}$$

$$\text{Sin } 210^o = \pm\frac{1}{2}$$

(ii) $$\text{Cos}\frac{\theta}{2} = \pm\sqrt{\frac{1+\text{Cos}\theta}{2}}$$

$$\text{Cos } 210^o = \pm\sqrt{\frac{1+\text{Cos } 420}{2}} = \pm\sqrt{\frac{1+\text{Cos } 60^o}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{1}{2}}{2}} = \pm\sqrt{\frac{\frac{3}{2}}{2}} = \pm\frac{\sqrt{3}}{\sqrt{4}}$$

$$\text{Cos } 210^o = \pm\frac{\sqrt{3}}{2}$$

(iii) $$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\cos 420^o}{1+\cos 420^o}} \quad (\theta = 420^o)$$

$$\tan\frac{420^o}{2} = \sqrt{\frac{1-\cos 60^o}{1+\cos 60^o}} = \sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}}$$

$$\tan 210^o = \sqrt{\frac{\frac{2-1}{2}}{\frac{2+1}{2}}} = \sqrt{\frac{1}{2}\times\frac{2}{3}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

Example 4:

Prove that $\dfrac{\text{Sin } 2\text{ A}}{\text{Sin A}} - \dfrac{\text{Cos } 2\text{ A}}{\text{Cos A}} = \text{Sec A}$

Solution:

$$\text{L.H.S} = \frac{\text{Sin } 2\text{ A}}{\text{Sin A}} - \frac{\text{Cos } 2\text{ A}}{\text{Cos A}}$$

$$= \frac{\text{Sin } 2\text{ A Cos A} - \text{Cos } 2\text{ A Sin A}}{\text{Sin A Cos A}}$$

$$= \frac{\text{Sin } (2\text{A} - \text{A})}{\text{Sin A Cos A}} = \frac{\text{Sin A}}{\text{Sin A Cos A}}$$

$$= \frac{1}{\text{Cos A}}$$

= Sec A

Exercise 4.2

Q.1 If $\text{Cos }\theta = \dfrac{4}{5}$ and the terminal ray of θ is in the first quadrant find the value of

(i) $\text{Sin}\dfrac{\theta}{2}$ (ii) $\text{Cos}\dfrac{\theta}{2}$ (iii) $\tan\dfrac{\theta}{2}$

Q.2 If $\text{Sin}\,\theta = \dfrac{4}{5}$ and the terminal ray of θ is in the first quadrant, find the value of

(i) $\text{Sin}\,2\theta$ (ii) $\text{Cos}\,2\theta$

Q.3 If $\text{Cos}\,\theta = -\dfrac{5}{13}$ and the terminal side of θ is in the second quadrant, find the value of

(i) $\text{Sin}\dfrac{\theta}{2}$ (ii) $\text{Cos}\dfrac{\theta}{2}$

Q.4 If $\tan\,\theta = -\dfrac{1}{5}$, the terminal ray of θ lies in the second quadrant, then find:

(i) $\text{Sin}\,2\,\theta$ (ii) $\text{Cos}\,2\,\theta$

Prove the following identities:

Q.5 $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$

Q.6 $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

Q.7 $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Q.8 $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

Q.9 $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Q.10 $\dfrac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$

Q.11 $\sec 2A + \tan 2A = \dfrac{\cos A + \sin A}{\cos A - \sin A}$

Q.12 $\text{cosec}\,2\theta + \cot 2\theta = \cot\theta$

Q.13 $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

Q.14 $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

Q.15 $\dfrac{\cot^2\theta-1}{\operatorname{cosec}^2\theta}=\cos 2\theta$

Q.16 $\cos^4\theta-\sin^4\theta=\dfrac{1}{\sec 2\theta}$

Q.17 $\left(\sin\dfrac{\theta}{2}+\cos\dfrac{\theta}{2}\right)^2=1+\sin\theta$

Q.18 $(\sin\theta-\cos\theta)^2=1-\sin 2\theta$

Q.19 $\sin^4\theta=\dfrac{3}{8}-\dfrac{1}{2}\cos 2\theta+\dfrac{1}{8}\cos 4\theta$

Q.20 Compute the value of $\sin\dfrac{\pi}{12}$ from the function of $\dfrac{\pi}{6}$

Answers 4.2

Q.1 (i) $\dfrac{1}{\sqrt{10}}$ (ii) $\dfrac{3}{\sqrt{10}}$ (iii) $\dfrac{1}{3}$

Q.2 (i) $\dfrac{24}{25}$ (ii) $-\dfrac{7}{25}$

Q.3 (i) $\dfrac{3}{\sqrt{13}}$ (ii) $\dfrac{2}{\sqrt{13}}$

Q.4 (i) $-\dfrac{5}{13}$ (ii) $\dfrac{12}{13}$

Q.21 $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

4.8 Conversion of sum or difference to products:

We know that

$$\mathrm{Sin}(\alpha+\beta)=\mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \beta+\ \mathrm{Cos}\ \alpha\ \mathrm{Sin}\ \beta \ \ldots\ldots \ (1)$$

$$\mathrm{Sin}(\alpha-\beta)=\mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \beta-\mathrm{Cos}\ \alpha\ \mathrm{Sin}\ \beta \ \ldots\ldots \ (2)$$

$$\mathrm{Cos}(\alpha+\beta)=\mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \beta-\mathrm{Sin}\ \alpha\ \mathrm{Sin}\ \beta \ \ldots\ldots \ (3)$$

$$\mathrm{Cos}(\alpha-\beta)=\mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \beta+\mathrm{Sin}\ \alpha\ \mathrm{Sin}\ \beta \ \ldots\ldots \ (4)$$

Adding (1) and (2), we get

$$\mathrm{Sin}(\alpha+\beta)+\mathrm{Sin}(\alpha+\beta)=2\mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \beta \qquad \ldots\ldots \ (5)$$

Subtracting (2) from (1)

$$\mathrm{Sin}(\alpha+\beta)-\mathrm{Sin}(\alpha-\beta)=2\mathrm{Cos}\ \alpha\ \mathrm{Sin}\ \beta \qquad \ldots\ldots \ (6)$$

Adding (3) and (4), we have

$$\mathrm{Cos}(\alpha+\beta)+\mathrm{Cos}(\alpha-\beta)=2\mathrm{Cos}\ \alpha\ \mathrm{Cos}\beta \qquad \ldots\ldots \ (7)$$

Subtracting (4) from (3), we have

$$\mathrm{Cos}(\alpha+\beta)-\mathrm{Cos}(\alpha-\beta)=-2\mathrm{Sin}\ \alpha\ \mathrm{Sin}\ \beta \qquad \ldots\ldots \ (8)$$

With the help of (5), (6), (7) and (8), we have get another set of important formulas

Let $\alpha+\beta=\mathrm{A}$ and $\alpha-\beta=\mathrm{B}$

Adding these, we have

$$2\ \alpha=\mathrm{A}+\mathrm{B} \qquad \Rightarrow \alpha=\frac{\mathrm{A}+\mathrm{B}}{2}$$

Subtracting these, we have

$$2\ \beta=\mathrm{A}-\mathrm{B} \qquad \Rightarrow \beta=\frac{\mathrm{A}-\mathrm{B}}{2}$$

Now putting these values of α and β in formulas from (5) to (8), we get

$$\mathrm{Sin\ A}+\mathrm{Sin\ B}=2\ \mathrm{Sin}\frac{\mathrm{A}+\mathrm{B}}{2}\mathrm{Cos}\frac{\mathrm{A}-\mathrm{B}}{2} \qquad (9)$$

$$\mathrm{Sin\ A}-\mathrm{Sin\ B}=2\ \mathrm{Cos}\frac{\mathrm{A}+\mathrm{B}}{2}\mathrm{Sin}\frac{\mathrm{A}-\mathrm{B}}{2} \qquad (10)$$

$$\mathrm{Cos\ A}+\mathrm{Cos\ B}=2\ \mathrm{Cos}\frac{\mathrm{A}+\mathrm{B}}{2}\mathrm{Cos}\frac{\mathrm{A}-\mathrm{B}}{2} \qquad (11)$$

$$\mathrm{Cos\ A}-\mathrm{Cos\ B}=-2\ \mathrm{Sin}\frac{\mathrm{A}+\mathrm{B}}{2}\mathrm{Sin}\frac{\mathrm{A}-\mathrm{B}}{2} \qquad (12)$$

4.9 Converting Products to Sum or Difference:

If we write the formulas given in (5) to (8) in reverse order, we have

$$2\ \mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \beta=\ \mathrm{Sin}(\alpha+\beta)+\mathrm{Sin}(\alpha-\beta) \qquad (13)$$

$$2\ \text{Cos}\ \alpha\ \text{Sin}\ \beta = \text{Sin}(\alpha+\beta) - \text{Sin}(\alpha-\beta) \qquad (14)$$
$$2\ \text{Cos}\ \alpha\ \text{Cos}\ \beta = \text{Cos}(\alpha+\beta) + \text{Cos}(\alpha-\beta) \qquad (15)$$
$$2\ \text{Sin}\ \alpha\ \text{Sin}\ \beta = \text{Cos}(\alpha-\beta) - \text{Cos}(\alpha+\beta) \qquad (16)$$

The formulas from (13) to (16) express products into sum or difference.

Example 1:

Express $\text{Sin}8\,\theta + \text{Sin}4\,\theta$ as products.

Solution:

We use the formula

$$\text{Sin A} + \text{Sin B} = 2\ \text{Sin}\ \frac{A+B}{2}\ \text{Cos}\frac{A-B}{2}$$

$$\text{Sin}8\,\theta\ \text{Sin}4\,\theta = 2\ \text{Sin}\frac{8\theta+4\theta}{2}\ \text{Cos}\frac{8\theta-4\theta}{2}$$

$$= 2\ \text{Sin}\frac{12\,\theta}{2}\ \text{Cos}\frac{4\,\theta}{2}$$

$$\text{Sin}8\,\theta\ \text{Sin}4\,\theta = 2\ \text{Sin}6\,\theta\ .\ \text{Cos}2\,\theta$$

Example 2:

Express $\text{Sin}\ \theta + \text{Sin}3\ \theta + \text{Sin}5\ \theta + \text{Sin}7\ \theta$ as a product.

Solution:

$$\text{Sin}\ \theta + \text{Sin}3\ \theta + \text{Sin}5\ \theta + \text{Sin}7\ \theta$$
$$= (\text{Sin}7\ \theta + \text{Sin}\ \theta) + (\text{Sin}5\ \theta + \text{Sin}3\ \theta)$$
$$= 2\ \text{Sin}\left(\frac{7\theta+\theta}{2}\right)\ \text{Cos}\left(\frac{7\theta-\theta}{2}\right) + 2\text{Sin}\left(\frac{5\theta+3\theta}{2}\right)\text{Cos}\left(\frac{5\theta-3\theta}{2}\right)$$
$$= 2\ \text{Sin}\frac{8\,\theta}{2}\ \text{Cos}\frac{6\,\theta}{2} + 2\ \text{Sin}\frac{8\,\theta}{2}\text{Cos}\frac{2\,\theta}{2}$$
$$= 2\text{Sin}4\,\theta\ \text{Cos}\ 3\,\theta + 2\ \text{Sin}4\,\theta\ \text{Cos}\,\theta$$
$$= 2\text{Sin}4\,\theta\ [\text{Cos}\ 3\,\theta + \text{Cos}\,\theta]$$
$$= 2\ \text{Sin}4\,\theta\left[2\ \text{Cos}\frac{3\,\theta+\theta}{2}\text{Cos}\frac{3\,\theta-\theta}{2}\right]$$
$$= 2\ \text{Sin}4\,\theta\left[2\ \text{Cos}\frac{4\,\theta}{2}\text{Cos}\frac{2\,\theta}{2}\right]$$
$$= 2\ \text{Sin}4\,\theta\ [2\ \text{Cos}2\,\theta\ \text{Cos}\,\theta]$$
$$= 4\ \text{Sin}4\,\theta\ \text{Cos}2\,\theta\ \text{Cos}\,\theta$$

Example 3:

Prove that $\text{Sin}\ 10^o\ \text{Sin}\ 30^o\ \text{Sin}\ 50^o\ \text{Sin}\ 70^o = \dfrac{1}{16}$

Solution:

L.H.S $= \text{Sin } 10^o \text{ Sin } 30^o \text{ Sin } 50^o \text{ Sin } 70^o$
$= \text{Sin } 30^o \text{ Sin } 10^o \text{ Sin } 50^o \text{ Sin } 70^o$
$= \frac{1}{2}[\text{Sin } 10^o \text{ Sin } 50^o] \text{ Sin } 70o$ because $\text{Sin } 30^o = \frac{1}{2}$
$= \frac{1}{4}[2 \text{ Sin } 10^o \text{ Sin } 50^o] \text{ Sin } 70^o$

Since, $2 \text{ Sin}\alpha \text{ Sin}\beta = \text{Cos}(\alpha - \beta) - \text{Cos}(\alpha + \beta)$

$$= \frac{1}{4}\left[\text{Cos}(10^o - 50^o) - \text{Cos}(10^o + 50^o)\right]\text{Sin}70^o$$
$$= \frac{1}{4}\left[\text{Cos}(-40^o) - \text{Cos } 60^o\right]\text{Sin}70^o$$
$$= \frac{1}{4}\left[\text{Cos}40^o - \frac{1}{2}\right]\text{Sin}70^o = \frac{1}{4}\left[\frac{2\text{Cos}40^o - 1}{2}\right]\text{Sin}70^o$$
$$= \frac{1}{8}\left[2\text{Sin}70^o \text{ Cos}40^o - \text{Sin}70^o\right]$$

We know $2 \text{ Sin}\alpha \text{ Cos}\beta = \text{Sin}(\alpha + \beta) + \text{Sin}(\alpha - \beta)$

$$= \frac{1}{8}\left[\text{Sin}(70^o + 40^o) + \text{Sin}(70^o - 40^o) - \text{Sin } 70^o\right]$$
$$= \frac{1}{8}\left[\text{Sin } 110^o + \text{Sin}30^o - \text{Sin}70^o)\right]$$
$$= \frac{1}{8}\left[\text{Sin}(180^o - 70^o) + \frac{1}{2} - \text{Sin}70^o\right]$$
$$= \frac{1}{8}\left[\text{Sin } 70^o + \frac{1}{2} - \text{Sin}70^o\right] = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16} = \text{R.H.S}$$

Example 4:

Prove that $\dfrac{\text{Sin A + 2Sin3A + Sin5A}}{\text{Sin3A + 2Sin5A + Sin7A}} = \dfrac{\text{Sin3A}}{\text{Sin5A}}$

Solution:

$$\text{L.H.S} = \frac{\text{Sin A + 2Sin3A + Sin5A}}{\text{Sin3A + 2Sin5A + Sin7A}}$$
$$= \frac{\text{(Sin5A + Sin A) + 2Sin3A}}{\text{(Sin7A + Sin3A) + 2Sin5A}}$$

$$= \frac{2\text{Sin}\frac{5A+A}{2}\text{Cos}\frac{5A-A}{2}+2\text{Sin}3A}{2\text{Sin}\frac{7A+3A}{2}\text{Cos}\frac{7A-3A}{2}+2\text{Sin}5A}$$

$$= \frac{2\text{Sin}3A\ \text{Cos}\ 2A + 2\text{Sin}3A}{2\text{Sin}5A\ \text{Cos}2A + 2\text{Sin}5A}$$

$$= \frac{2\text{Sin}3A\ (\text{Cos}\ 2A + 1)}{2\text{Sin}5A\ (\text{Cos}2A + 1)}$$

$$= \frac{\text{Sin}3A}{\text{Sin}5A} = \text{R.H.S}$$

Exercise 4.3

Q.1 Express each of the following sum or difference as products.

(i) $\text{Sin}5\,\theta - \text{Sin}\,\theta$ (ii) $\text{Cos}\,\theta - \text{Cos}5\,\theta$

(iii) $\text{Cos}12\,\theta - \text{Cos}4\,\theta$ (iv) $\text{Sin}\frac{50}{3} - \text{Sin}\frac{50}{6}$

(v) $\text{Cos}\left(\frac{\alpha+\beta}{2}\right) + \text{Cos}\left(\frac{\alpha-\beta}{2}\right)$ (vi) $\text{Sin}4\,\theta + \text{Sin}2\,\theta$

Q.2 Express each of the following products as sum or difference.

(i) $2\ \text{Sin}3\,\theta\ \text{Cos}\,\theta$ (ii) $\text{Sin}3\,\theta\ .\ \text{Cos}5\,\theta$

(iii) $\text{Cos}3\,\theta\ .\ \text{Cos}5\,\theta$ (iv) $\text{Sin}\frac{\alpha+\beta}{2}\text{Cos}\frac{\alpha-\beta}{2}$

Q.3 Express $\text{Sin}3\,\theta + \text{Sin}5\,\theta + \text{Sin}7\,\theta + \text{Sin}9\,\theta$ as a product.

Prove the following identities:

Q.4 $\frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan\theta$ Q5. $\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta} = -\cot\theta$

Q.6 $\frac{\text{Cos}\,\beta + \text{Cos}\,9\beta}{\text{Sin}\,\beta + \text{Sin}\,9\beta} = \cot 5\beta$ Q.7 $\frac{\text{Sin}\,3\theta - \text{Sin}\,\theta}{\text{Cos}^2\theta - \text{Sin}^2\theta} = 2\text{Sin}\,\theta$

Q.8 $\frac{\text{Sin}\,A + \text{Sin}\,B}{\text{Cos}\,A + \text{Cos}\,B} = \tan\frac{A+B}{2}$

Q.9 $\frac{\text{Cos}\,2\theta - \text{Cos}\,6\theta}{\text{Cos}\,2\theta + \text{Cos}\,6\theta} = \tan 4\theta\tan 2\theta$

Q.10 $\dfrac{\text{Cos } \alpha - \text{Cos } \beta}{\text{Cos } \alpha + \text{Cos } \beta} = -\dfrac{\tan\left(\dfrac{\alpha+\beta}{2}\right)}{\cot\left(\dfrac{\alpha-\beta}{2}\right)}$

Q.11 $\dfrac{\text{Sin } \theta + \text{Sin } 2\theta + \text{Sin } 3\theta}{\text{Cos } \theta + \text{Cos } 2\theta + \text{Cos } 3\theta} = \tan 2\theta$

Q.12 $\text{Sin } 5\theta + 2 \sin 3\theta + \sin \theta = 4 \sin 3\theta \cos^2\theta$

Q.13 Show that:

(i) $\dfrac{\text{Sin } 75^o - \text{Sin } 15^o}{\text{Cos } 75^o + \text{Cos } 15^o} = \dfrac{1}{\sqrt{3}}$

(ii) $\text{Sin } 20^o + \text{Sin } 40^o = \text{Cos } 10^o$

(iii) $\text{Cos } 80^o + \text{Cos } 40^o = \text{Cos}20^o$

(iv) $\text{Cos } 20^o + \text{Cos } 100^o + \text{Cos } 140^o = 0$

Prove that.

Q.14 $\text{Sin } 20^o \text{ Sin } 40^o \text{ Sin } 60^o \text{ Sin } 80^o = \dfrac{3}{16}$

Q.15 $\text{Sin } 10^o \text{ Sin } 30^o \text{ Sin } 50^o \text{ Sin } 70^o = \dfrac{1}{16}$

Q.16 $\text{Sin } 20^o \text{ Sin } 40^o \text{ Sin } 80^o \text{ Sin } 90^o = \dfrac{\sqrt{3}}{8}$

Q.17 $\text{Cos } 20^o \text{ Cos } 40^o \text{ Cos } 60^o \text{ Cos } 80^o = \dfrac{1}{16}$

Answer 4.3

Q.1 (i) $2 \text{ Cos } 3\theta \text{ Sin } 2\theta$ (ii) $2 \text{ Sin } 3\theta \text{ Sin } 2\theta$

(iii) $2 \text{ Cos } 8\theta \text{ Cos } 4\theta$ (iv) $2 \text{ Cos } \dfrac{150}{12} \text{ Sin} \dfrac{50}{12}$

(v) $2 \text{ Cos } \dfrac{\alpha}{2} \text{ Cos} \dfrac{\beta}{2}$ (vi) $2 \text{ Sin } 3\theta \text{ Cos } \theta$

Q.2 (i) $\text{Sin } 4\theta + \text{Sin } 2\theta$ (ii) $\dfrac{1}{2}[\text{Sin } 8\theta - \text{Sin } 2\theta]$

(iii) $\dfrac{1}{2}[\text{Cos } 8\theta + \text{Cos } 2\theta]$ (iv) $\dfrac{1}{2}[\text{Sin}\alpha + \text{Sin}\beta]$

Q.3 $4\text{Cos } \theta \text{ Sin } 6\theta \text{ Cos } 2\theta$

Short Questions

Write the short answers of the following:

Q.1 Prove that: $\text{Cos}(-\beta) = \text{Cos}\beta$

Q.2 Prove that: $\sin(-\theta) = -\sin\theta$

Q.3 Prove that: $\tan(-\theta) = -\tan\theta$

Q.4 $\text{Cos}\left(\frac{\pi}{2} - \beta\right) = \sin\beta$

Q.5 Prove that $\text{Sin}\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Q.6 $\text{Sin}(\pi + \theta) = -\sin\theta$

Q.7 Show that: $\text{Sin}(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

Q.8 $\text{Cos}(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$

Q.9 Prove that: $\text{Sin}\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

Q.10 Prove that: $\tan(45^0 + \theta)\tan(45^0 - \theta) = 1$

Q.11 Express: $\sin x \cos 2x - \sin 2x \cos x$ as single term

Q.12 Express: $\text{Cos}(a+b)\cos(a-b) - \sin(a+b)\sin(a-b)$ as single term.

Q.13 Prove that: $\text{Cos}\,2\alpha = \cos^2\alpha - \sin^2\alpha$

Q.14 Prove that: $\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

Q.15 Prove that: $\text{Sin}^2\alpha = \frac{1 - \cos 2\alpha}{2}$

Q.16 Prove that: $\text{Cos}^2\alpha = \frac{1 + \text{Cos}\,2\alpha}{2}$

Q.17 If $\sin\theta = \frac{4}{5}$ and the terminal side of θ lies in 1^{st} quadrant,find $\cos\frac{\theta}{2}$

Q.18 Prove that: $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

Q.19 Prove that: $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$

Q.20 Express the sum as product: $\cos 12\theta + \cos 4\theta$

Q.21 Express $\cos\theta - \cos 4\theta$ as product:

Q.22 Express as sum or difference $2\cos 5\theta \sin 3\theta$

Q.23 Express as sum or difference $\cos 3\theta \cos\theta$

Q.24 Express $\sin(x + 30^0) + \sin(x - 30^0)$ as product

Q.25 Find $\cos\theta$ if $\sin\theta = \frac{7}{25}$ and angle θ is an acute angle**.**

Answers

Q.11 $-\sin x$ Q.12 $\cos 2a$ Q.17 $\frac{2}{\sqrt{5}}$

Q.20 $2\cos 8\theta \cos 4\theta$ Q.21 $2\sin\frac{5\theta}{2}\sin\frac{3\theta}{2}$

Q.22 $\sin 8\theta - \sin 2\theta$ Q.23 $\frac{1}{2}[\cos 4\theta + \cos 2\theta]$

Q.24 $2\sin x \cos 30^0$ Q.25 $\frac{24}{25}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Sin $(\alpha+\beta)$ is equal to:

(a) Sin α Cos $\beta +$ Cos α Sin β

(b) Cos α Cos $\beta -$ Sin α Sin β

(c) Sin α Cos $\beta -$ Cos α Sin β

(d) Cos α Cos $\beta +$ Sin α Sin β

__2. Cos $(\alpha-\beta)$ is equal to:

(a) Cos α Cos $\beta -$ Sin α Sin β

(b) Cos α Cos $\beta +$ Sin α Sin β

(c) Cos α Sin $\beta -$ Sin α Cos β

(d) Sin α Cos $\beta +$ Cos α Sin β

__3. tan $(45^o - x)$ is equal to:

(a) $\dfrac{\text{Cos x + Sin x}}{\text{Cos x} - \text{Sin x}}$ (b) $\dfrac{1+\tan x}{1-\tan x}$

(c) $\dfrac{1+\cot x}{1-\cot x}$ (d) $\dfrac{\text{Cos x} - \text{Sin x}}{\text{Cos x + Sin x}}$

__4. $\cos\left(\dfrac{\pi}{2}+\theta\right)$ is equal to:

(a) Cos θ (b) –Cos θ

(c) Sin θ (d) –Sin θ

__5. Sin (90o – 0) is equal to:

(a) –Sin θ (b) Sin θ

(c) –Cos θ (d) Cos θ

__6. $\text{Sin}(\pi - x)$ is equal to:

(a) –Sin x (b) Sin x

(c) Cos x (d) –Cos x

__7. $\tan\left(\dfrac{\pi}{2}+\theta\right)$ is equal to:

(a) Tan θ (b) Cot θ

(c) – Cot θ (d) – Tan θ

__8. $\text{Cos}(\pi+\theta)$ is equal to:

(a) Cos θ (b) –Sin θ

(c) –Cos θ (d) Sin θ

__9. $\text{Cos}\left(\frac{\pi}{2}+\theta\right)$ is equal to:

(a) Cos θ (b) Sin θ
(c) –Cos θ (d) –Sin θ

__10. $\tan\left(\frac{3\pi}{2}+\theta\right)$ is equal to:

(a) Tan θ (b) − Tan θ
(c) Cot θ (d) − Cot θ

__11. $\frac{\text{Sin}(\alpha+\beta)}{\text{Cos }\alpha\text{ Sin }\beta}$ is equal to:

(a) $\tan\alpha-\tan\beta$ (b) $\tan\alpha+\tan\beta$
(c) $\text{Sin }\alpha+\text{Sin }\beta$ (d) $\text{Sin }\alpha-\text{Sin }\beta$

__12. $\text{Sin }2\alpha$ is equal to:

(a) $\text{Cos}^2\alpha-\text{Sin}^2\alpha$ (b) $\text{Cos }2\alpha$
(c) $1-\text{Cos}^2\alpha$ (d) $2\text{Sin }\alpha\text{ Cos}\alpha$

__13. $2\text{Cos}^2\frac{\theta}{2}$ is equal to:

(a) $1+\text{Cos}\theta$ (b) $1-\text{Cos}\theta$
(c) $1+\text{Sin }\theta$ (d) $1-\text{Sin }\theta$

__14. $\text{Cos}(\alpha+\beta)-\text{Cos}(\alpha-\beta)$ is equal to:

(a) $2\text{ Sin }\alpha\text{ Cos }\beta$ (b) $2\text{ Cos }\alpha\text{ Sin }\beta$
(c) $2\text{ Cos }\alpha\text{ Cos }\beta$ (d) $-2\text{ Sin }\alpha\text{ Sin }\beta$

__15. Cos A – Cos B is equal to:

(a) $2\text{Cos}\frac{A+B}{2}\text{Cos}\frac{A-B}{2}$ (b) $-2\text{Sin}\frac{A+B}{2}\text{Sin}\frac{A-B}{2}$

(c) $2\text{Sin}\frac{A+B}{2}\text{Cos}\frac{A-B}{2}$ (d) $2\text{Cos}\frac{A+B}{2}\text{Sin}\frac{A-B}{2}$

__16. Sin (A + B) – Sin (A –B) is equal to:

(a) 2 Sin A Cos B (b) 2 Cos A Cos B
(c) –2 Sin A Sin B (d) 2 Cos A Sin B

__17. Cos (A – B) – Cos (A+B) is equal to:

(a) 2 Sin A Sin B (b) –2 Sin A Sin B
(c) 2 Cos A Cos B (d) 2 Cos A Sin B

__18. $\text{Sin }5\theta-\text{Sin }2\theta$ is equal to:

(a) 2Sin 3θ Cos 2θ (b) 2Cos 3θ Sin 2θ
(c) 2Cos 3θ Cos 2θ (d) –2Cos 3θ Sin 2θ

__19. $\text{Sin } 5\theta + \text{Sin } \theta$ is equal to:

(a) $2\text{Sin } 3\theta \text{ Cos } 2\theta$ (b) $-2\text{Cos } 3\theta \text{ Sin } 2\theta$

(c) $2\text{Cos } 3\theta \text{ Sin } 2\theta$ (d) $2\text{Sin } 3\theta \text{ Sin } 2\theta$

__20. $2\text{Sin } 6\theta \text{ Cos } 2\theta$ is equal to:

(a) $\text{Sin } 8\theta + \text{Sin } 4\theta$ (b) $\text{Sin } 8\theta - \text{Sin } 4\theta$

(c) $\text{Cos } 8\theta + \text{Cos } 4\theta$ (d) $\text{Cos } 8\theta - \text{Cos } 4\theta$

Answers

1.	a	2.	b	3.	d	4.	d	5.	d
6.	b	7.	c	8.	c	9.	d	10.	d
11.	b	12.	d	13.	a	14.	d	15.	a
16.	d	17.	a	18.	c	19.	a	20.	a