Chapter 12

The Straight Line

(Plane Analytic Geometry)

12.1 Introduction:

Analytic- geometry was introduced by Rene Descartes (1596 – 1650) in his La Geometric published in 1637. Accordingly, after the name of its founder, analytic or co-ordinate geometry is often referred to as Cartesian geometry. It is essentially a method of studying geometry by mean of algebra. Its main purpose was to show how a systematic use of coordinates (real numbers) could vastly simplify geometric arguments. In it he gave a simple technique of great flexibility for the solution of a variety of problems.

12.2 Rectangular Coordinates:

Consider two perpendicular lines X′X and Y′Y intersecting point in the point O (Fig. 1). X'X is called the x-axis and Y'Y the y-axis and together they form a rectangular coordinate system. The axes divide the plane into four **quadrants** which are usually labeled as in trigonometry. The point O is called the origin, When numerical scales are established on the axes, positive distances x (abscissa) are drawn to the right of the origin, negative distance to the left; positive distance y (ordinates) are drawn upwards and negative distances downwards to the origin. Thus OX and OY have positive direction while OX′, OY′ have negatives direction.

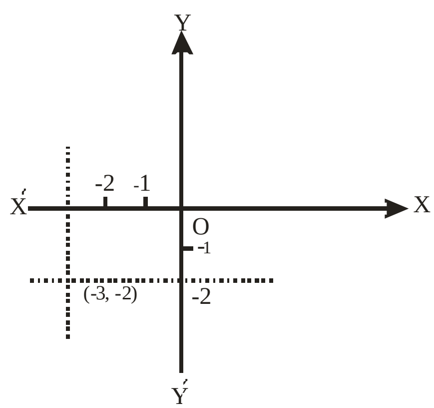


Fig 12.1

We now consider an arbitrary point P in the plane and the lines through P parallel to the axes. (These parallel lines might coincide with an axis if P is on the axis). The line through P parallel to the y-axis will intersect the x-axis at a point corresponding to some real number a. This number is called the x-coordinate (or abscissa) of P. the line through P parallel to the x-axis will intersect the y-axis at a point corresponding to some real number b. this number is called the y-coordinate (or ordinate) of P. The real numbers a and b are the coordinates of P and we indicate the point and coordinates by P (a, b) or by (a, b). In the Fig.1 the point P (-3,-2) is platted frequently. We shall refer to the order pair of real numbers (a, b) as a point. The coordinates a and b of a point (a, b) are called the Rectangular coordinates or Cartesian coordinates.

12.3 Polar Coordinates:

The Cartesian coordinates that we have been using specify the location of a point in the plane by giving the directed distances of the point from a pair of fixed perpendicular lines, the axes. There is an alternative coordinate system that is frequently used in the plane, in which the location of a point is specified in a different way.

In a plane, consider a fixed ray PB and any point A(Fig. 2) we can describe the location of A by giving the distance r from p to A and specifying the angle θ (measured in degree or radian). By stating the order pair (r, θ) we clearly identify the location of A. The components of such an ordered pair are celled polar coordinates of A. The fixed ray PB is called the polar axis and the initial point of the polar axis is called the pole of the system.

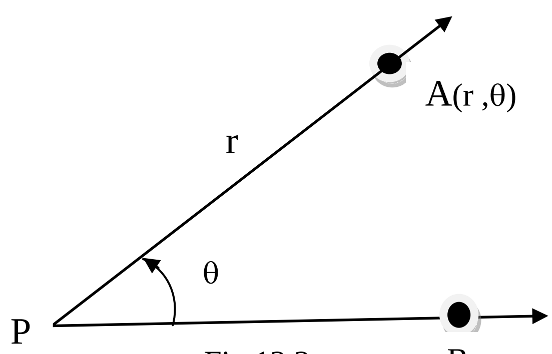


Fig.12.2

In polar coordinate system each point in the plane has infinitely many pair of polar coordinates. In the first place, if (r, θ) are polar coordinate of A, then so are (r, $\theta + K360^o$ for each $k \in j$ (Fig 3a). In the second place, if we let – r < 0 denote the directed distance from P to A along the negative extension of the ray PA' in the direction opposite that of PA (Fig. 3b). Then we see that also $(-r, \theta + 180^o)$ and more generally $(-r, \theta + 180° + K360°)$, $K \in J$, are polar coordinates of A'. The pole P itself represented by $(0, \theta)$ for any θ whatsoever.

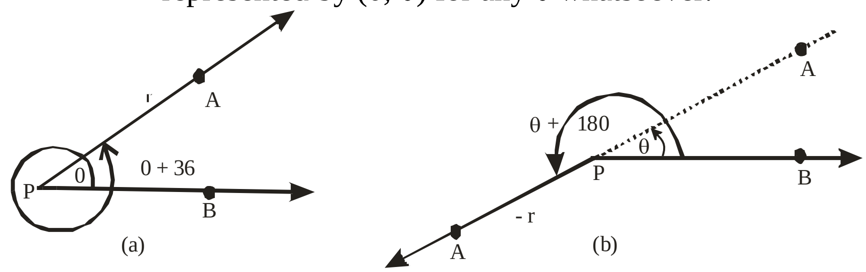


Fig 12,3

12.4 Relation between Rectangular and Polar Coordinates:

If the pole P in a rectangular coordinates system is also the origin O in a Cartesian coordinates system, and if the polar axis coincides with the positive x-axis of the cartesian system, then the coordinates (x, y) can be expressed in terms of the polar coordinates (r, θ) by the following equations (Fig. 4).

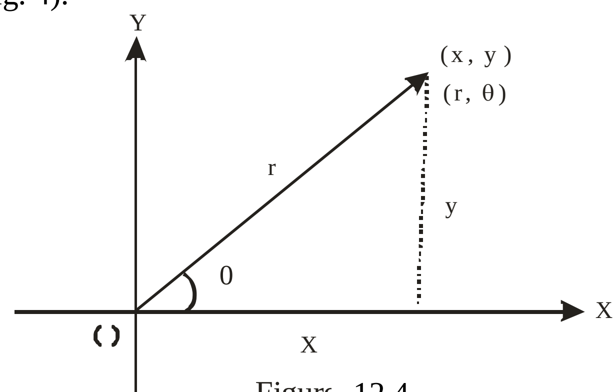


Figure 12.4

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

conversely, we have

$$r = \pm\sqrt{x^2 + y^2}$$

and $$\tan\theta = \frac{y}{x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Or, $\theta = \tan^{-1}\frac{y}{x}$

The sets of equations (1) and (2) enable us to find rectangular coordinates for a point when given a pair of polar coordinates and vice versa.

Example 1:

Find the rectangular coordinates of the point with polar coordinates (4, 30^o)

$$x = r\cos\theta \quad = 4\cos 30^o \quad = 4\,\frac{\sqrt{3}}{2} \quad = 2\sqrt{3}$$

$$y = r\sin\theta \quad = 4\sin 30^o \quad = 4\,\frac{1}{2} \quad = 2$$

The rectangular coordinates are $(2\sqrt{3}, 2)$

Example 2:

Find a pair of polar coordinates for the point with cartesian coordinates (7, -2).

Solution:

$$r = \sqrt{x^2 + y^2} = \sqrt{49 + 4} = \sqrt{53}$$

Noting that (7, -1) is in the fourth quadrant, so

$$\text{Tan}\,\theta = \frac{y}{x} = \frac{-2}{7}$$

$$\theta = \text{tan-1}\left(-\frac{2}{7}\right) = -16^o$$

12.5 The Distance Formula (distance between two points):

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points lying in the first quadrant. Let d be the distance between the points P and Q. Draw PR and QR parallel to the coordinates axes (Fig.12.5). By simple subtraction of abscissa, $PR = x_2 - x_1$; similarly subtracting ordinates, $QR = y_2 - y_1$.

Since PQR is a right triangle, so by Pythagorean theorem, we have.

$$(PQ)^2 = (PR)^2 + (QR)^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Or} \quad = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This is known as the **distance formula.** The same formula holds true regardless of the quadrants in which the points lie.

Note: The distance d is given positive, because we are interested to find the numerical value of d and not its direction.

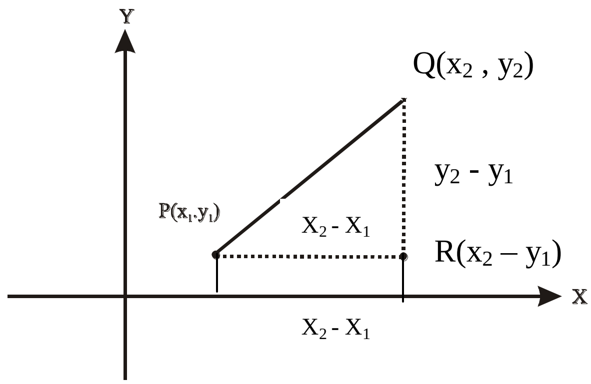


Figure 12.5

Example 3:

Find the distance between (–3, –2) and (–1,5).

Solution:

$$d = \sqrt{(-1+3)^2 + (5+2)^2}$$
$$= \sqrt{4+49} = \sqrt{53} = 7.42$$

Example 4:

Show that the points (–3, 1) (2,4) and (0, -4) are vertices of a right triangle.

Solution:

Let A (–3, 1), B(2,4) and C(0, –4), then

$$|AB| = \sqrt{(2+3)^2 + (4-1)^2} = \sqrt{25+9} = \sqrt{34} \text{ Units}$$
$$|BC| = \sqrt{(0-2)^2 + (-4-4)^2} = \sqrt{4+64} = \sqrt{68} \text{ Units}$$
$$\text{And } |AC| = \sqrt{(0+3)^2 + (-4-1)^2} = \sqrt{9+25} = \sqrt{34} \text{ Units}$$

since $|AB|^2 + |AC|^2 = |BC|^2$

So the given points are the vertices of a right triangle, with right angle at point A.

Example 5:

Show that the point (3, $\sqrt{7}$) is on a circle with centre at the origin and radius 4.

Solution:

Let A (O,O) and B (3, $\sqrt{7}$)

The distance between A and B is:

$|AB| = \sqrt{(3-0)^2 + (\sqrt{7}-0))^2} = \sqrt{9+7} = 4$

Which is the radius of the circle, so the given point (3, $\sqrt{7}$) lies on the circle.

Exercise 12.1

Q.1: Find the distance between:

(a) (–4,2) and (0,5) (b) (2, –2) and (2,7)

(c) (1–$\sqrt{2}$,1 –$\sqrt{3}$) and (1 + $\sqrt{2}$, 1 + $\sqrt{3}$

(d) (a, b) and (a + c, b + d)

Q.2: Show that the points

(a) A (2, 2), B(6, 6) and C(11,1) are the vertices of a right triangle.

(b) A(1, 0), B(–2, –3), C(2, –1) and D(5,2) are the vertices of a parallelogram.

(c) A (2, 3), B(0, –1) and C(–2, 1) are the vertices of an isoscles triangle.

Q.3: Is the point (0,4) inside or outside the circle of radius 4 with centre at (–3, 1) ?

Q.4: Determine y so that (0, y) shall be on the circle of radius 4 with centre at (–3, 1).

Q.5: The point (x, y) is on the x-axis and is 6 units away from the point (1, 4), find x and y.

Q.6: If one end of a line whose length is 13 Units is the point (4, 8) and the ordinate of the other end is 3. What is its abscissa?

Q.7: Find a point having ordinate 5 which is at a distance of 5 units from the point (2, 0).

Q.8: Find the value of y so that the distance between (1, y) and (–1, 4) is 2.

Q.9: Find the coordinates of the point that is equidistant from the points (2, 3), (0, –1) and (4, 5).

Q.10: Show that the points A(–3, 4), B(2,6) and C(0,2) are collinear. Find the values of AC: CB and AB: BC.

Answers

Q.1: **(a)** 5 **(b)** 9 **(c)** $2\sqrt{5}$ **(d)** $\sqrt{c^2 + d^2}$

Q.3: Outside **Q.4:** $1 \pm \sqrt{7}$

Q.5: $x = 1 \pm 2\sqrt{5}$, $y = 0$ **Q.6:** 16, –8 **Q.7:** (2,5)

Q.8: 4 **Q.9:** (11, –4)

Q.10: $3\sqrt{5} : 2\sqrt{5}$; $5\sqrt{5} : 2\sqrt{5}$

12.6 Segment of Line:

It is a part of a straight line between two points on it. The segment contains one end point or both end points. The sign of a line segment has a plus or minus sign according to some convention. Thus, if $P_1 P_2$ is the positive directed segment, then P_2P_1 has the negative sense and we write $P_1P_2 = – P_2P_1.$

12.7 The Ratio Formula (point of division):

Given a directed line segment such as P_1P_2 in Fig.6; to find the coordinates of the point P which divides internally P_1P_2 in a given ratio $r_1 : r_2$. Let the coordinates of points P, P_1, and P_2

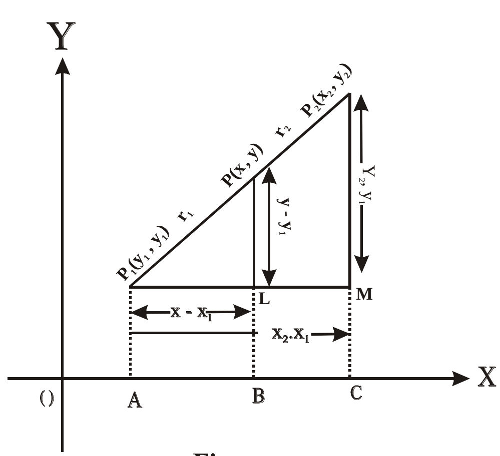


Figure 12.6

are (x, y), (x_1, y_1) and (x_2, y_2) respectively.

From points P_1, P and P_2 drawn P_1A, PB and P_2C perpendicular on x – axis. Also draw a line P_1M parallel to x-axis meeting PB at point L.

Now from the similar triangles P_1PL and P_1P_2M, we have

$$\frac{P_1L}{P_1M} = \frac{P_1P}{P_1P_2}$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{r_1}{r_1 + r_2}$$

$$x - x_1 = \frac{r_1(x_2 - x_1)}{r_1 + r_2}$$

$$x = x_1 + \frac{r_1(x_2 - x_1)}{r_1 + r_2}$$

Or $$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2};\ r_1 + r_2 \neq 0$$

Similarly, $$\frac{PL}{P_2M} = \frac{P_1P}{P_1P_2}$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{r_1}{r_1 + r_2}$$

$$y - y_1 = \frac{r_1(y_2 - y_1)}{r_1 + r_2}$$

$$y = y_1 + \frac{r_1(y_2 - y_1)}{r_1 + r_2}$$

Or $$y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2};\quad r_1 + r_2 \neq 0$$

Hence coordinates of point P are:

$$\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

Corollary 1:

If P divides P_1P_2 externally, then P_1P and PP_2 are measured in opposite directions. So in the ratio $r_1 : r_2$ either r_1 or r_2 is negative. The corresponding coordinates of P for external point, are obtained just by giving negative sign to either r_1 or to r_2. So far external division the coordinates of P are.

$$\left(\frac{r_1\,x_2 - r_2\,x_1}{r_1 - r_2}, \frac{r_1\,y_2 - r_2\,y_1}{r_1 - r_2}\right) \qquad , \qquad \text{if } r_1 > r_2$$

Or $$\left(\frac{r_2\,x_1 - r_2\,x_1}{r_1 - r_2}, \frac{r_1\,y_2 - r_2\,y_1}{r_2 - r_1}\right) \qquad , \qquad \text{if } r_2 > r_1$$

Corollary 2:

(Coordinate of Mid-point)

For the mid point P of the segment $P_1\,P_2$.

$r_1 = r_2 \Rightarrow$ 1 : 1. Therefore, the mid point P has the coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 7:

Find the coordinates of the mid point of the segment.

$P_1(3, 7)$, $P_2(-2, 3)$

Solution:

If (x, y) is the mid point of the segment, then

$$x = \frac{3 + (-2)}{2} = \frac{1}{2}$$

$$y = \frac{7 + 3}{2} = 5$$

$$P(x, y) = P\left(\frac{1}{2}, 5\right)$$

Example 8:

Find the coordinates of the point P which divides the segments P_1 (–2, 5), P_2(4, –1) in the ratio of

(a) $\frac{r_1}{r_2} = \frac{6}{5}$, (b) $\frac{r_1}{r_2} = -2$ (c) $\frac{r_1}{r_2} = -\frac{1}{3}$

Solution:

(a) $r_1 : r_2 = 6 : 5$

$$x = \frac{6(4) + 5(-2)}{6 + 5} = \frac{14}{11}$$

$$y = \frac{6(-1) + 5(5)}{6 + 5} = \frac{19}{11}$$

(b) $r_1 : r_2 = -2 : 1$

$$x = \frac{-2(4) + 1(-2)}{-2 + 1} = 10$$

$$y = \frac{-2(-1) + 1(5)}{-2 + 1} = -7$$

(c) $r_1 : r_2 = -1 : 3$

$$x = \frac{-1(4) + 3(-2)}{-1 + 3} = -5$$

$$y = \frac{-1(-1) + 3(5)}{-1 + 3} = 8$$

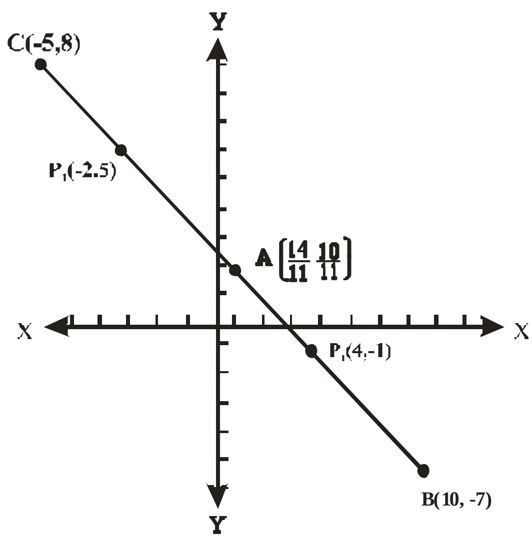


Figure 12.7

Example 9:

Find the ratio in which the line joining (–2, 2) and (4, 5)

is cut by the axis of y.

Solution:

Let the ratio be $r_1 : r_2$

$$x = \frac{4r_1 - 2r_2}{r_1 + r_2} \quad =$$

Since on y – axis, x = 0

So, $$0 = \frac{4r_1 - 2r_2}{r_1 + r_2}$$

Or $4r_1 - 2r_2 = 0$

Or $2r_1 = r_2$

Or $\frac{r_1}{r_2} = \frac{1}{2}$

Or $r_1 : r_2 = 1 : 2$

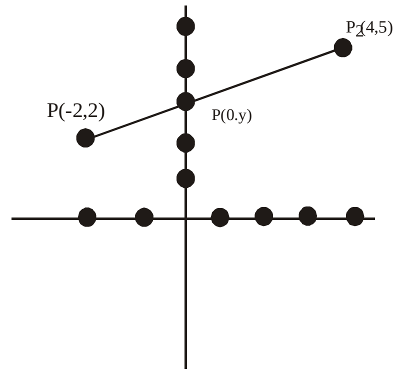
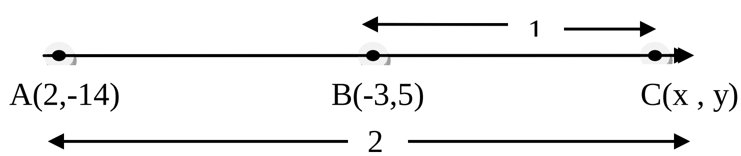


Figure 12.8

Example 10:

Find the point reached by going from the point (2, –14) to the point (–3, 5) and then proceeding an equal distance beyond the latter point.

Solution:



$r_1 : r_2 = 2 : 1$

Let P (x, y) be a point which is to find.

$$x = \frac{r_1 x_2 - r_2 x_1}{r_1 - r_2} = \frac{2(-3) - 1(2)}{2 - 1} = -8$$

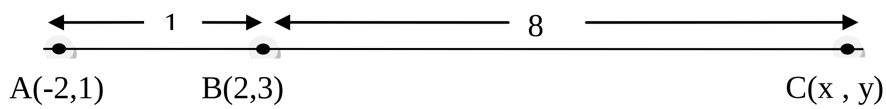
$$y = \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2} = \frac{2(5) - 1(-14)}{2 - 1} = 24$$

Hence $\quad P(x, y) = P(-8, 24)$

Example 11:

Let A (–2, 1), B(2, 3) and C (x, y) are collinear with B between A and C and if |BC| = 8|AB|,find the point C(x, y)

Solution:



Since $\quad |BC| = 8\,|AB|$

$$\frac{|BC|}{|AB|} = \frac{8}{1}$$

So $\quad |AC| : |CB| = 9 : 8 = r_1 : r_2$

Since. C(x, y) is external point, so by formula

$$x = \frac{r_1 x_2 - r_2 x_1}{r_1 - r_2} \quad , \quad y = \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2}$$

$$x = \frac{9(2) - 8(-2)}{9 - 8} \quad , \quad y = \frac{9(3) - 8(-1)}{9 - 8}$$

$$x = 34 \quad , \quad y = 35$$

Hence C (x, y) = C (34, 35)

Exercise 12.2

Q.1: Assuming that the points $P_1(2, 3)$, $P_2(4, 2)$ and $P_3(6, 1)$ are collinear, find the ratio $P_1P_2 : P_2P_3$.

Q.2: Obtain the ratio in which the point (3, – 2) divides the line joining the points (1, 4) and (– 3, 16).

Q.3: Find the point which is three fifth i.e. $\left(\frac{3}{5}\right)$ from the point (4,1) to the point (5,7).

Q.4: Find the point which is $\frac{7}{10}$ of the way from the point (4, 5) to the point (–6, 10).

Q.5: Find the point which is two third of the way from the point (5, 1) to the point (–2, 9)

Q.6: Let P(0, 4), Q (5, 0) and R(x, y) are collinear with P between R and Q and if |RP| = 10 |PQ|, find the coordinates of R(x, y).

Q.7: If A(–4, 2), B(6, –4) and C (x, y) are collinear with B between A and C and if |AC| = 5 |AB|, find the coordinates of C.

Q.8: Find the point of trisection of the median of the triangle with vertices at (– 1, –2), (4, 2) and (6, 3).

Q.9: Find the points trisecting the join of A(-1, 4) and B(6 , 2)

Q.10: Find the coordinates of the points that trisect the segment whose end points are (a, b) and (c, d).

Q.11: The mid points of the sides of a triangle are at (–1, 4), (5, 2) and (2, –1). Find its vertices.

Answers

Q.1: 1 : 1 internally **Q.2:** 1:3 externally

Q.3: $\left(\frac{23}{5}, \frac{23}{5}\right)$ **Q.4:** (–3, 8.5) **Q.5:**(1/3, 19/3)

Q.6: (–50, 44) **Q.7:** (46, –28) **Q.8:** (3, 1)

Q.9: (4/3 , 10/3) and (11/3 , 8/3)

Q.10: $\left(\frac{2a+c}{3}, \frac{2b+d}{3}\right); \left(\frac{a+2c}{3}, \frac{b+2d}{3}\right)$ **Q.11:** (–4, 1), (2,7)(8, –3)

12.8 Inclination and Slope of a Line:

The angle θ ($0 \leq \theta < 180^o$), measured counter clockwise from the positive x – axis to the line is called the inclination of the line or the angle of inclination of the line.

The tangent of this angle i.e. tan θ, is called the slope or gradient of the line. It is generally denoted by m. Thus $m = \tan\theta$, $0 \leq \theta < 180^o$. The slope of the line is positive or negative according as the angle of inclination is acute or obtuse.

Now $P_1 P_2 R$ is a right triangle,

Then $m = \tan\theta = \dfrac{P_1R}{P_2R}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Vertical cange}}{\text{Horizontal change}}$$

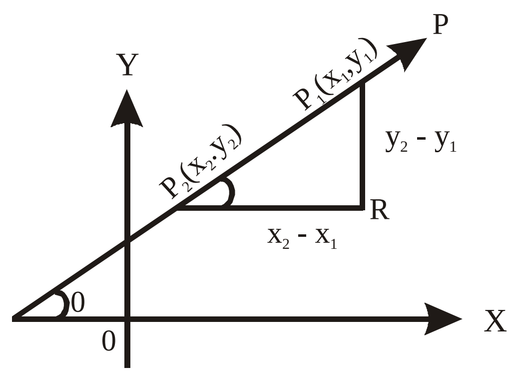


Figure 12.9

Note:

(i) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

(ii) The slope of a line parallel to x – axis is zero, because then $y_2 - y_1 = 0$.(fig.9)

(iii) The slope of x – axis is also zero.

(iv) The slope of a line parallel to y – axis is not defined, because then $x_2 - x_1 = 0$

(v) The slope of y – axis is also not defined.

12.9 Parallel and Perpendicular Lines:

The concept of slope is a convenient tool for studying parallel and perpendicular lines. The slopes of the vertical do not exist.

Theorem – I:

Two lines are parallel or coincide if and only if they have the same slope.

Proof:

Let ℓ_1 and ℓ_2 be two parallel lines. The inclinations of the lines are θ_1 and θ_2 respectively. Therefore

$$\theta_1 = \theta_2$$

Or $\tan\theta_1 = \tan\theta_2$

$m_1 = m_2$ i.e, the slopes of ℓ_1 and ℓ_2 are equal:

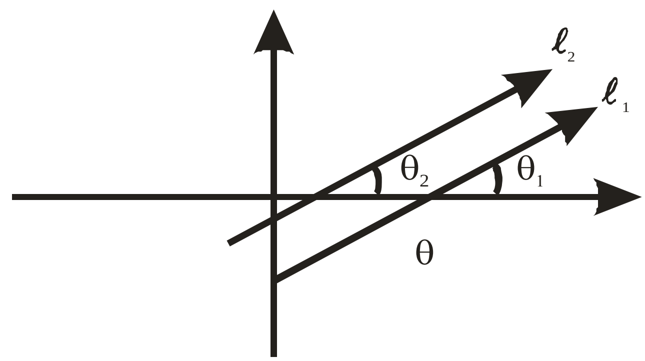


Figure 12.10

Conversely, if $m_1 = m_2$, then the two lines ℓ_1 and ℓ_2 are parallel.

Theorem 2:

Two lines are perpendicular if and only if the product of their slopes is –1.

Proof: Let ℓ_1 and ℓ_2 be two perpendicular lines with inclinations θ_1 and θ_2 respectively from fig.12.11

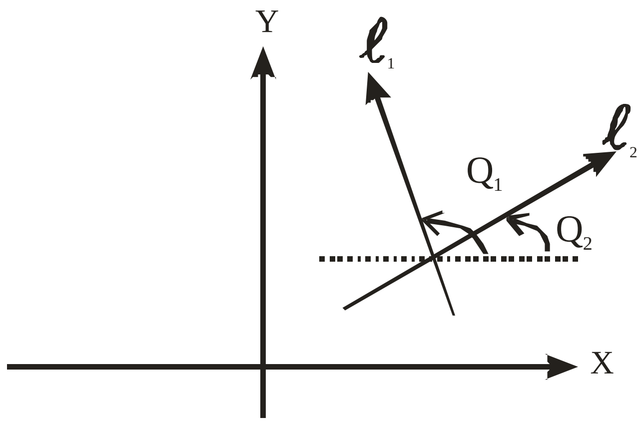


Figure 12.11

$$\theta_1 = 90^o + \theta_2$$

Or $\tan\theta_1 = \tan(90^o + \theta_2)$

Or $\tan\theta_1 = -\text{Cot}\,\theta_2$

Or $\tan\theta_1 = -\dfrac{1}{\tan\ \theta_2}$

Or $m_1 = -\dfrac{1}{m_2}$

Or $m_2\, m_1 = -1$

Conversely if $m_1m_2 = -1$, then the two lines ℓ_1 and ℓ_2 are perpendicular.

Example 12: Find the slope of a line which is perpendicular to the line joining P_1 (2, 4), P_2(–2, 1).

Solution:

The slope of the line $P_1\,P_2$ is

$$m_1 = \frac{1-4}{-2-2} = \frac{3}{4}$$

Therefore the slope of a perpendicular line is $m_2 = -\dfrac{4}{3}$

Example 13: Show that the points A(–5, 3), B(6, 0) and C(5, 5) are the vertices of a right triangle.

Solution:

$$\text{Slope of AB} = m_1 = \frac{0-3}{6+5} = -\frac{3}{11}$$

$$\text{Slope of BC} = m_2 = \frac{5-0}{5-6} = -5$$

$$\text{Slope of AC} = m_3 = \frac{5-3}{5+5} = \frac{1}{5}$$

Since $m_2\, m_3 = -1$, so the sides AC and BC are perpendicular, with vertices C at right angle. Hence the given points are the vertices of a right triangle.

12.10 Angle Between Two Lines:

If θ is the angle between two lines ℓ_1 and ℓ_2 then from Fig.12

$\theta \Rightarrow \theta_2 - \theta_1$

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

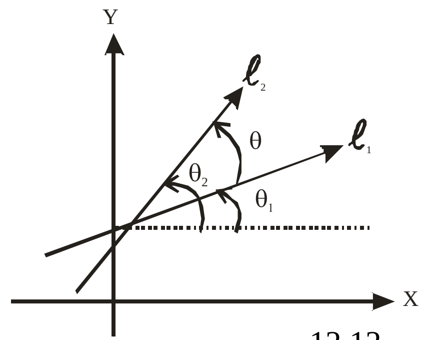


Figure 12.12

Example 14:

Find the angle from the line with slope $-\frac{7}{3}$ to the line with slope $\frac{5}{2}$

Solution:

Here $m_1 = -\frac{7}{3}$, $m_2 = \frac{5}{2}$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = = \frac{\frac{5}{2} - \left(-\frac{7}{3}\right)}{1 + \frac{5}{2}\left(-\frac{7}{3}\right)} = \frac{29}{-29} = -1$$

$$\theta = \tan^{-1}(-1) = 135^0$$

Example 15:

Show that the points (2 , 6) , (-8 , 1) and (-2 , 4) are collinear.

Solution:

Let the given points be A(2 , 6) , B (-8 , 1) and C (-2 , 4), then

$$\text{Slope of line AB} = \frac{1-6}{-8-2} = \frac{-5}{-10} = \frac{1}{2}$$

$$\text{Slope of line AC} = \frac{4-6}{-2-2} = \frac{-2}{-4} = \frac{1}{2}$$

∴ A , B and C are collinear

Exercise 12.3

Q.1: Show that the two lines passing through the given points are perpendicular.

(a) (0, –7), (8, – 5) and (5, 7), (8, – 5)

(b) (8, 0), (6, 6) and (–3, 3) , (6, 6)

Q.2: If a line ℓ_1 contains P (2, 6) and (0, y). Find y if ℓ_1 is parallel to ℓ_2 and that the slope of $\ell_2 = \frac{3}{4}$

Q.3: For the triangle A (1, 3), B(–2, 1), C(0, – 4), find

(a) Slope of a line perpendicular to AB.

(b) Slope of a line parallel to AC.

(c) Angle ABC

Q.4: Show that the given points are the vertices of a right triangle.

(a) (0, 6), (9, – 6) and (–3, 0)

(b) (1, –1), $\left(-\frac{39}{25}, 7\right)$ and $\left(\frac{29}{4}, 1\right)$

Q.5: Show that the given points are the vertices of a parallelogram.

(a) (–3, 1), (–1, 7), (2, 8) and (0, 2)

(b) (1, 0), (–2, –3), (2, –1) and (5, 2)

Q.6: Find the slopes of the sides and altitudes of the triangles whose vertices are the points (2, 3), (0, –1) and (–2, 1).

Q.7: Show that the points (2, 6), (–8, 1) and (–2, 4) are collinear by using slope.

Answers

Q.2: $\frac{9}{2}$ **Q.3**: (a) $-\frac{3}{2}$ (b) 7, (c) $\tan\theta = -\frac{19}{4}$

Q.6: $2, -1, \frac{1}{2}; -\frac{1}{2}, 1, -2$

12.11 Equation of a Straight Line:

Straight Line:

The line, in Euclidean geometry, which passes through two points in such a way that the length of the segment between the points is a minimum. Or the straight line is a curve with constant slope.

The equation of a line is an equation in x, y which is satisfied by every point of the line.

12.11.1When Parallel to X – Axis:

Let ℓ be a line parallel to x – axis at a distance of ‘b’ units. Then the equation of the line ℓ is the locus of the point P (x, y) which moves such that it remains at a constant distance b units from the x – axis i.e., y coordinate of P is always equal to b. therefore.

y = b is the required equation, for example the equation of the line passing through (1,4)(2, 4) (3, 4) etc. is y = 4 or y – 4=0

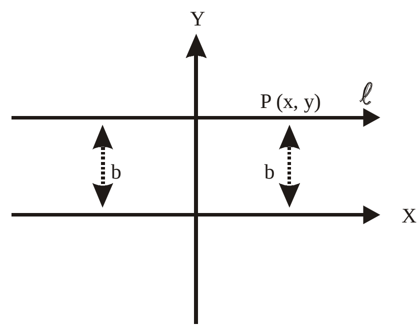


Figure 12.13

12.11.2. When parallel to Y – axis:

Let ℓ be a line parallel to Y – axis at a distance of ‘a’ units. Then the equation of the line ℓ is the locus of the point P (x, y) which moves such that it remains at a constant distance a units from the Y – axis i.e., X-Coordinates of P is always equal to a. Therefore.

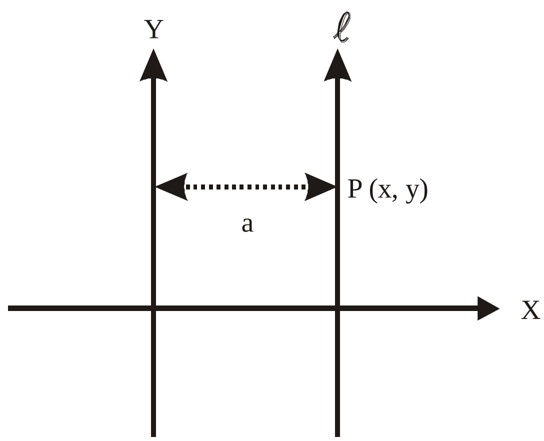


Figure 12.14

x = a is the required equation, for example the equation of the line passing through (3, 1), (3, 3) (3, – 5) etc. is x = 3 or x – 3 = 0

12.12 Three Important Forms of the Equation of a Line:

12.12.1 Point – slope form:

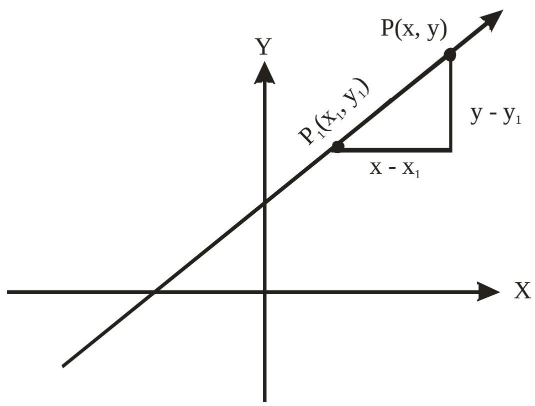
Suppose a line having slope m and passing through a given point (x_1, y_1) as shown in Fig. 15. If P(x, y) is any other point on the line, then the slope of the line is

$$\frac{y - y_1}{x - x_1} = m$$

From which $y - y_1 = m(x - x_1)$

This is called point slope form for a linear equation.

(1)



Figu 12.15

Corollary I:

Slope – Intercept Form:

Suppose a line having slope m, passing through a given point on the y – axis having coordinates (0, c) as show in Fig.16 substituting (0, C) in the point slope form of a linear equation.

$y - y_1 = m(x - x_1)$

We obtain $y - c = m(x - 0)$

From which $y = mx + c$

This equation is called slope – intercept form.

(2)

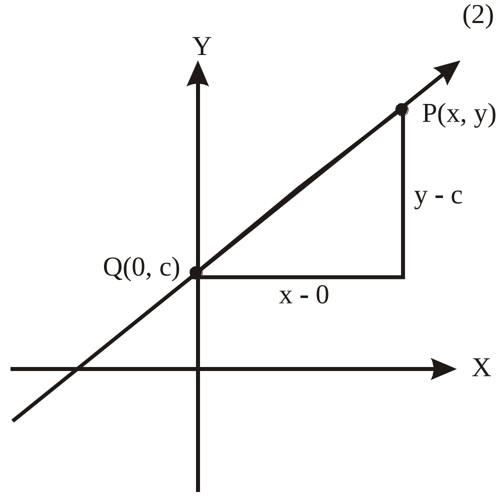


Figure 8.16

Where, m is the slope and C is the y- Intercept
Equation of the line passing through the origin is $y = mx$.

Corollary – II:

Two Point Form:

Suppose a line is passing through the points (x_1, y_1) and (x_2, y_2), then slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad 12.16$$

Using point slope form of a linear equation

$$y - y_1 = m(x - x_1)$$

We obtain $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad (3)$$

This is the two point form of the linear equation.

Example 14: Find the equation of the line passing through the point (–2, 3) and having slope $-\frac{1}{2}$

Solution:

By point – slope form, its equation is

$$y - 3 = -\frac{1}{2}(x + 2)$$

Or $2y - 6 = -x - 2$

Or $x + 2y - 4 = 0$

Example 15: Find an equation of the line with slope $-\frac{2}{3}$ and having y- intercept 3.

Solution:

By sloping-intercept form, we have

$$y = -\frac{2}{3}x + 3$$

Or $3y = -2x + 9$

Or $2x + 3y - 9 = 0$

Example 16: Write the equation, in standard form, of the line with the same slope as 2y – 3x = 5 and passing through (0, 5).

Solution:

Write the equation $2y - 3x = 5$ in slope- intercept form

$$y = \frac{3}{2}x + \frac{5}{2}\ , \quad \text{the slope } m = \frac{3}{2}$$

Now by point-slope form (Or slope- intercept form)

We have, $y - 5 = \frac{3}{2}\ (x - 0)$

$2y - 10 = 3x$

Or $3x - 2y + 10 = 0$

Example 17: Find the equation of the line through (–1, 2) and (3, –4)

Solution:

Equation of the line through two point is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

So $\frac{y - 2}{-4 - 2} = \frac{x + 1}{3 + 1}$

Or $\frac{y - 2}{-6} = \frac{x + 1}{4}$

Or $6x + 4y - 2 = 0$

Or $3x + 2y - 1 = 0$

OR Alternatively, slope of (–1, 2) and (3, –4) is

$$m = \frac{-4 - 2}{3 + 1} = \frac{-6}{4} = \frac{-3}{2}$$

Now by point-slope form, point (–1, 2), (Or (3, –4))

We have $y - 2 = -\frac{3}{2}\ (x + 1)$

$2y - 4 = -3x - 3$

Or $3x + 2y - 1 = 0$

12.12.2 Intercept Form:

Suppose a and b are the x and y-intercepts of a straight line of point A and B respectively. Then the coordinates of point A and B are (a,0) and (0, b) respectively.

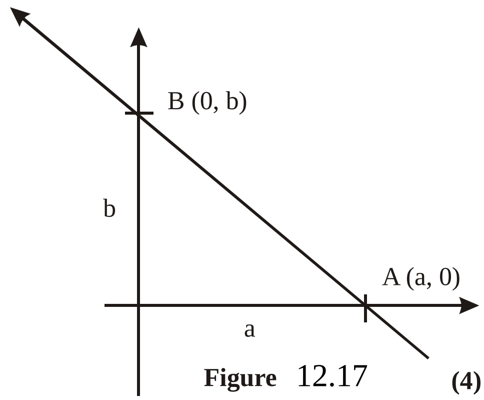


Figure 12.17 **(4)**

Therefore the slope of AB.

$$m = \frac{b-0}{0-a} = -\frac{b}{a}$$

Using point-slope from

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{b}{a}(x - a)$$

Divide both sides by b,

Or $$\frac{y}{b} = -\frac{x}{a} + 1$$

Or $$\frac{x}{a} + \frac{y}{b} = 1$$

This is called intercept form of a linear equation.

Example 18: Find the equation of the line passing through (–8,–1) and making equal intercepts on the coordinate axes.

Solution:

Here a = b, so equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1$$

Putting (x, y) = (–8, –1), we have

$$\frac{-8}{a} + \frac{-1}{a} = 1$$

$$\frac{-9}{a} = 1$$

$$a = -9$$

$$\frac{x}{-9} + \frac{y}{-9} = 1$$

Or $$x + y + 9 = 0$$

Is the required equation.

Example 19: A line makes the positive intercepts on coordinate axes whose sum is 7. It passes through (– 3, 8) find the equation.

Solution:

If a and b are the positive intercepts, then

$$a + b = 7$$

Or $$b = 7 - a$$

By the Equation $$\frac{x}{a} + \frac{y}{b} = 1$$

We have $$\frac{x}{a} + \frac{y}{7-a} = 1$$

As this line is passing through the point (–3, 8), so

$$\frac{-3}{a} + \frac{8}{7-a} = 1$$

$$-21 + 3a + 8a = 7a - a^2$$

Or $$a^2 + 4a - 21 = 0$$

By solving this equation we get,

$$a = 3 \quad \text{or} \quad a = -7$$

Since a is positive, so $a = 3$

and $b = 4$

Hence $$\frac{x}{3} + \frac{y}{4} = 1$$

$$4x + 3y = 12$$

Or $4x + 3y - 12 = 0$ is the required equation.

12.12.3 Perpendicular or Normal Form:

Suppose P is the perpendicular length from the origin O to the point A on the line ℓ and θ is the angle of inclination of perpendicular P. The equation of the line ℓ which is passing through the point A can be found in terms of P and θ.

The coordinates of A are ($p \cos \theta$, $p \sin \theta$).

The slope of AO is tan θ, since the line ℓ is perpendicular to OA, so slope of line ℓ is $-\dfrac{1}{\tan\theta} = -\cot\theta$

Using point slope form.

$$y - y_1 = m(x - x_1)$$

We have, $y - p\sin\theta = -\cot\theta\,(x - p\cos\theta)$

Or $y - p\sin\theta = -\dfrac{\cos\theta}{\sin\theta}(x - p\cos\theta)$

Or $y\sin\theta - p\sin^2\theta = -x\cos\theta + p\cos^2\theta$

Or $x\cos\theta + y\sin\theta = p$

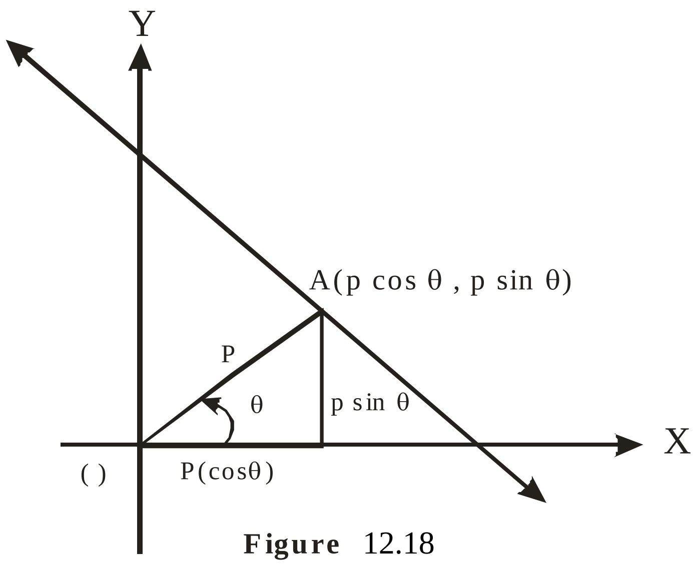


Figure 12.18

This is the perpendicular or Normal form of a linear equation.

Example 20: Find the equation of the line when θ = 45° and $p=\dfrac{1}{\sqrt{2}}$.

Solution:

Since the equation of the Normal form is

$$x \cos\theta + y \sin\theta = p$$

Putting $\theta = 45^o$, and $p = \frac{1}{\sqrt{2}}$

$$x \cos 45^o + y \sin 45^o = \frac{1}{\sqrt{2}}$$

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Or $x + y = 1$ or $x + y - 1 = 0$ is the required equation.

Exercise 12.4

Q.1: Find equation for the lines:

(a) through (4 , 2) and (–5, –1)

(b) through (–1 , –1) with slope $-\frac{1}{2}$

(c) through $\left(-\frac{7}{3} \quad 0\right)$ and $\left(-\frac{5}{2} \quad 0\right)$

(d) through (–1, –2) and parallel to y – axis.

Q.2: Find the slope and y – intercept:

(a) $ax + by = b, b \neq 0$ (b) $\sqrt{2}\, x + (1 - \sqrt{2})\, y = 2$

Q.3: Determine the real number k so that the two lines $5x - 3y = 12$ and $kx - y = 2$ will be

(a) parallel (b) Perpendicular

Q.4: Show that the given points are collinear:

(a) (1, 0), (–4, – 12) and (2, –4)

(b) (– 4,4), (– 2,1) and (6, – 11)

Q.5: Find the equations of the medians of the triangle with vertices (–4, –6), (0, 10), (4, 2).

Q.6: Find the equations of the three altitudes of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

Q.7: A triangle has vertices at (0, 0), (a, b) and (c, d). Show that a line containing the mid points of the two of the sides of the triangle is parallel to the third side.

Q.8: Find the equation of the line which is perpendicular to the line x + 2y = 7 and having y – intercept 3.

Q.9: A line is parallel to the line 2x + 3y = 5 and passes through (–1, 3). Find an equation for the line.

Q.10: Write an equation of the line parallel to 2x – 7y = 8 and containing the origin.

Q.11: Find the line which is perpendicular to the line 4x + 7y = 5 and which passes through (–1, 2).

Q.12: What is an equation of a line perpendicular to 5x – y = 4 and containing the point (2, 3)?

Q.13: Find the equation of the line passing through (–1, 7) and perpendicular to the line through the points (2, 3) and (o, – 4).

Q.14: What are the x and y-intercepts of 3x + 4y = 12?

Q.15: Find the equation of the line whose intercept on x- axis is three times its intercept on y-axis and which passes through the point (–1, 3).

Q.16: A line makes the negative intercepts on the coordinates axes whose sum is –10. It passes through (4, –9), find its equation.

Q.17: Find the equation of the perpendicular bisector of the line segment joining the points.

(a) (2, 4) and (6, 8) (b) (–4, 6) and (6, 10)

(c) 3, –2) and (5, 4)

Answers

Q.1: (a) $x - 3y + 2 = 0$ (b) $x + 2y + 3 = 0$

(c) $y = 0$ (d) $x = -1$

Q.2: (a) $m = \frac{-a}{b}$; y – intercept = 1

(b) $m = -\frac{\sqrt{2}}{1 - \sqrt{2}}$; y – intercept = $\frac{2}{1 - \sqrt{2}}$

Q.3: (a) $k = \frac{5}{3}$ (b) $-\frac{3}{5}$

Q.5: Equations of the medians : $x = 0$, $y = 2$ and $2x - y = -2$

Q.6: $(y_2 - y_1)\, y + (x_2 - x_1)\, x = y_3\,(y_2 - y_1) + x_3(x_2 - x_1)$,

$(y_3 - y_2)\, y + (x_3 - x_2)\, x = y_1\,(y_3 - y_2) + x_1\,(x_3 - x_2)$,

$(y_3 - y_1)\, y + (x_3 - x_1)\, x = y_2\,(y_3 - y_1) + x_2\,(x_3 - x_1)$

Q.8: $2x - y + 3 = 0$ **Q.9**: $2x + 3y = 7$

Q.10: $2x - 7y = 0$ **Q.11:** $7x - 4y + 15 = 0$

Q.12: $x + 5y = 17$ **Q.13:** $2x + 7y - 47 = 0$

Q.14: $a = 4$, $b = 3$ **Q.15:** $x + 3y - 8 = 0$

Q.16: $x + y + 5 = 0$

Q.17: **(a)** $x + y = 10$ **(b)** $5x - 8y = 21$

(c) $3x - y + 11 = 0$

12.13 The General Linear Equation:

A linear equation in x and y is an equation of the form

$$Ax + By + C = 0 \qquad (1)$$

Where A and B are given real numbers and A and B are not both zero. The equation (1) is called the General linear equation because the graph of such on equation is always a straight line.

Theorem:

Every linear equation has a graph which is a straight line.

Proof:

Suppose we have a linear equation (first degree) in variables x and y.

$$Ax + By + C = 0$$

(i) If $B = 0$, then $A \neq 0$ and the equation is

$$Ax + C = 0$$

Or $$x = -\frac{C}{A}$$

The graph of this equation is a line parallel to the y – axis

(ii) If $B \neq 0$, then $A = 0$ and the equation is

$$By + C = 0$$

Or $$y = -\frac{C}{B}$$

The graph of this equation is a line parallel to the x – axis.

(iii) If $A \neq 0$, $B \neq 0$, then equation can be written in the form.

$$y = -\frac{A}{B}x - \frac{C}{B}$$

Again the graph of this equation is a straight line with slope $m = -\frac{A}{B}$ and y – intercept $c = -\frac{C}{B}$

Therefore, in all cases, the linear equation $Ax + By + C = 0$ represents a straight line.

12.14 <u>Reduction of General form Ax + By + C = 0 to other forms.</u>

(i) Reduction to slope – Intercept Form:

$$Ax + by + C = 0$$

Reduce to, $y = -\frac{A}{B}x - \frac{C}{B}$

Which is of the form $y = mx + c$

(ii) Reduction to Intercept Form:

$$Ax + By + C = 0$$

Or $Ax + By - C$

Or $\frac{A}{-C} + \frac{B}{-C} = 1$

Or $\frac{\frac{x}{-C}}{A} + \frac{\frac{y}{-C}}{A} = 1$

Which is of the form $\frac{x}{a} + \frac{y}{b} = 1$

(iii) Reduction to Perpendicular Form:

Comparing the equation

$$Ax + By + C = 0 \qquad (1)$$

With the perpendicular form

$$\text{Cos } \alpha.\ x + \sin \alpha\ y - P = 0 \qquad (2)$$

We have, $\frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = -\frac{C}{P} = k$

Because the co-efficients of (1) and (2) are proportional.

So $A = k \cos \alpha$, $B = k \sin \alpha$ and $C = -pk$ or $p = -\frac{C}{k}$

$$A^2 + B^2 = k^2 (\cos^2 \alpha + \sin^2 \alpha)$$

Or $A^2 + B^2 = k^2$

Or $k = \pm\sqrt{A^2 + B^2}$

So $p = -\left(\frac{C}{\pm\sqrt{A^2 + B^2}}\right)$

As p must be positive, so sign of C and $\sqrt{A^2 + B^2}$ are opposite.

If C is positive then $k = -\sqrt{A^2 + B^2}$

Therefore $\cos\alpha = \frac{A}{K} = -\frac{A}{\sqrt{A^2 + B^2}}$

$\cos\alpha = \frac{B}{K} = -\frac{B}{\sqrt{A^2 + B^2}}$

and $p = -\left(\frac{C}{-\sqrt{A^2 + B^2}}\right) = \frac{C}{\sqrt{A^2 + B^2}}$

Putting these values in eq. (2), we get

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y - \frac{C}{\sqrt{A^2 + B^2}} = 0$$

Or $$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

If C is negative, then $k = \frac{A}{\sqrt{A^2 + B^2}}$

So, $\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2 + B^2}}$, $p = -\frac{C}{\sqrt{A^2 + B^2}}$

Putting these values in equation (2), we get.

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

Hence the equation Ax + by + C = 0 is reduced in perpendicular form by dividing $+\sqrt{A^2 + B^2}$ or $-\sqrt{A^2 + B^2}$ accordingly C is negative or positive respectively.

Example 22: Reduce the equation 3x + 4y = 10 to the
(i) Slope-intercept from (ii) Intercept form
(ii) Normal form

Solution: (i)

$$3x + 4y = 10$$
$$4y = -3x + 10$$
$$y = -\frac{3}{4}x + \frac{5}{2}$$

Which is the slope-intercept form with slope $m = -\frac{3}{4}$, y – intercept = $C = \frac{5}{2}$.

(iii) $3x + 4y = 10$

$$\frac{3x}{10} + \frac{4y}{10} = 1$$

Or $$\frac{x}{\frac{10}{3}} + \frac{x}{\frac{5}{2}} = 1$$

Which is the intercepts form $\frac{x}{a} + \frac{y}{b} = 1$

With x – intercept $a = \frac{10}{3}$ and y – intercept $b = \frac{5}{2}$

(iii) $3x + 4y = 10$

$$k = \sqrt{A^2 + B^2} = \sqrt{9 + 16} = 5$$

Divide the equation by 5 $\Rightarrow \frac{3}{5}x + \frac{4}{5}y = 2$

Which is the perpendicular form

$$X \cos \alpha + y \sin \alpha = p$$

With $\cos \alpha = \frac{3}{5}$, $\sin \alpha = \frac{4}{5}$ and $p = 2$

$$\tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\frac{4}{3}$$

Hence the given line is at a distance of 2 unit from the origin and is perpendicular from the origin on the line with angle of inclination $\alpha = \tan^{-1}\frac{4}{3}$.

12.15 Intersection of Two Lines:

Let $a_1x + b_1y + c_1 = 0$ (1)

$a_2x + b_2y + c_2 = 0$ (2)

be the two lines. Their point of intersection can be obtained by solving them simultaneously.

$$\frac{x}{b_1c_2 - b_2c_1} = -\frac{y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

Hence point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\right)$$

If $a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ then the above coordinates have no meaning and the lines do not intersect but are parallel.

Hence the lines (1) and (2) intersect if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

12.16 Concurrent Lines and Point of Concurrency:

Three or more than three lines are said to be concurrent if these are intersecting at the same point. The point of intersection of these lines is called point of concurrency.

12.16.1 Condition of Concurrency of Three Lines:

Suppose the three line are

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

$$a_2x + b_2y + c_2 = 0 \qquad (2)$$

$$a_3x + b_3y + c_3 = 0 \qquad (3)$$

These lines will be concurrent if the point of intersection of any two lines satisfies the third line.

The point of intersection of (2) and (3) is

$$\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} \quad , \frac{a_3c_2 - a_2c_3}{a_2b_3 - a_3b_2}\right)$$

Putting this point in equation (1)

$$a_1\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}\right) + b_1\left(\frac{a_3c_2 - a_2c_3}{a_2b_3 - a_3b_2}\right) + c_1 = 0$$

$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2) = 0$

This equation can be written in the determinant form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example 23: Show that the three lines

$x - y + 6 \quad = 0$ ……………………….(1)

$2x + y - 5 \quad = 0$ ……………………….(2)

$-x - 2y + 11 \quad = 0$ ……………………….(2)

are concurrent. Also find the point of concurrency

Solution:

Since, $\begin{vmatrix} 1 & -1 & 6 \\ 2 & 1 & -5 \\ -1 & -2 & 11 \end{vmatrix}$

$= 1\,(11 - 10) + 1\,(22 - 5) + 6\,(-4 + 1)$

$= 1 + 17 - 18 = 0$

Hence the lines are concurrent.

For the point of concurrency, solving equations (1) and (2)
Adding equations (1) and (2), we get

$3x + 1 = 0$

Or $x = -\frac{1}{3}$, Put in (1)

$$y = \frac{1}{3} + 6 = \frac{19}{3}$$

So point of concurrency is $\left(-\frac{1}{3} \quad \frac{19}{3}\right)$

Example 24: Find K so that the lines

x – 2y + 1 = 0, 2x – 5y + 3 = 0 and 5x + 9y + k = 0 are concurrent.

Solution:

Since the lines are concurrent, so

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & -5 & 3 \\ 5 & 9 & k \end{vmatrix} = 0$$

$$1(-5k - 27) + 2(2k - 15) + 1(18 + 25) = 0$$

$$-5k - 27 + 4k - 30 + 43 = 0$$

$$-k - 14 = 0$$

Or $k = -14$

12.16.2 Condition that Three Points be Collinear:

The three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) will be collinear

If $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_3 - y_2}{x_3 - x_2}$

Or $(y_2 - y_1)(x_3 - x_2) = (y_3 - y_2)(x_2 - x_1)$

Or $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example 25: Show that the three points (1, 2), (7, 6), (4, 4) are collinear.

Solution:

Since $\begin{vmatrix} 1 & 2 & 1 \\ 7 & 6 & 1 \\ 4 & 4 & 1 \end{vmatrix} = 0$

$= 1\,(6 - 4) - 2(7 - 4) + 1\,(28 - 24)$

$= 2 - 6 + 4 = 0$

Therefore the points are collinear i.e., these lie on a line.

Exercise 12.5

Q.1: Reduce the given equations to

(i) Slope-intercept form (ii) Intercepts form
(iii) Normal form

(a) $3x + y = y^2$ (b) $6x - 5y = 15$

(c) $\sqrt{3} + \sqrt{6}\,y = \sqrt{2}$

Q.2: Determine p and α for the lines:
(a) $x - 5y + 3 = 0$ (b) $x + y + 3\sqrt{2} = 0$

Q.3: Show that the following lines are concurrent. Also find the point of concurrency.

(a) $3x - 5y + 8 = 0$, $x + 2y - 4 = 0$ and $4x - 3y + 4 = 0$

(b) $2x - 3y - 7 = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$

(c) $5x + y + 11 = 0$, $x + 7y + 9 = 0$ and $2x + y + 5 = 0$

Q.4: Find K so that $x + y + 1 = 0$

$kx - y + 3 = 0$

and $4x - 5y + k = 0$ will be concurrent.

Q.5: Show that the altitudes of the triangle whose vertices are (–1, 2), (4, 3) and (1, –2) intersect at a point. Find the coordinates of the point of intersection.

(Hint: First find the equations of the altitudes, then show them concurrent).

Q.6: Find the equations of the medians of the triangle with vertices (–4, –6), (0, 10), (4, 2). Show that the medians meet in a point.

Q.7: Show that the given points are collinear

(a) (–4, 4), (–2,1) and (6, –11)

(b) (1, 9), (–2, 3) and (–5, –3)

Q.8: Find the value of k so that (1, –3), (–2, 5), (4, k) lie on a line.

Answers

Q.1: (a) (i) $y = -3x - 2$ (ii) $\frac{x}{-2/3} + \frac{y}{-2} = 1$

(iii) $\frac{-3}{2}x - \frac{y}{2} = 1$

(b) (i) $y = \frac{6}{5}x - 3$ (ii) $\frac{x}{15/6} + \frac{y}{-3} = 1$

(iii) $\frac{6}{\sqrt{61}}x - \frac{5}{\sqrt{61}}y, = \frac{15}{\sqrt{61}}$

(c) (i) $y = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{3}}$ (ii) $\frac{x}{\sqrt{\frac{2}{3}}} + \frac{y}{\sqrt{\frac{1}{3}}} = 1$

(iii) $\frac{1}{\sqrt{3}}x + \sqrt{\frac{2}{3}}\,y = \frac{\sqrt{2}}{3}$

Q.2: (a) $p = \frac{3}{\sqrt{26}}$, $\alpha = 92^{o}\,34'$ (b) $p = 3$, $\alpha = 225^{o}$

Q.3: (a) $\left(\frac{4}{11}, \frac{20}{11}\right)$ (b) (11,5) (c) (–2, 1)

Q.4: $k = -3 \pm 2\sqrt{10}$

Q.5: $\left(\frac{4}{11}, \frac{13}{11}\right)$ is the point of intersection.

Q.6: Equation of the medians : $x = 0$, $y = 2$ and $2x - y = -2$. The medians intersect at (0, 2).

Q.8: $k = -11$

12.17 The Distance from a Point to a Line:

To find the distance from a point $P_1 (x_1, y_1)$ to the line $Ax + By + C = 0$, draw $P_1 P_2 = d$ perpendicular on the line. The coordinates of P_2 are (x_2, y_2).

By distance formula

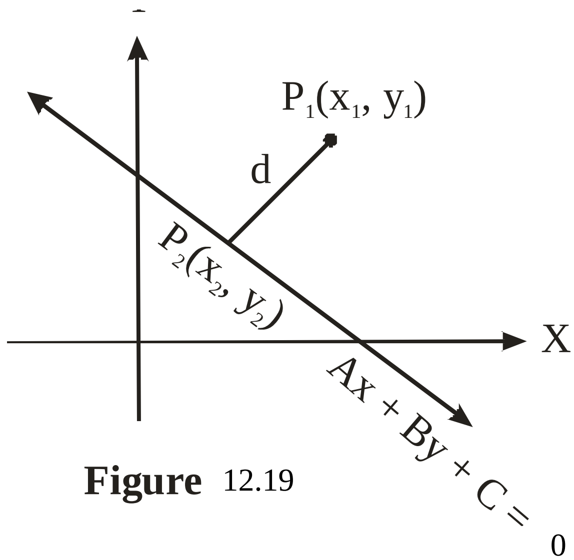


Figure 12.19

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (1)$$

We find the points $P_2 (x_2, y_2)$

Since point $P_2 (x_2, y_2)$ lies on the line $Ax + By + C = 0$

So $\quad Ax_2 + By_2 + C = 0 \qquad (2)$

From equation $Ax + By + C = 0$

$$y = -\frac{A}{B}x - \frac{C}{B}$$

The slope of lines is $-\frac{A}{B}$

The slope of perpendicular $\overline{P_1P_2}$ is $m = \dfrac{B}{A}$

The equation of perpendicular $\overline{P_1P_2}$ passing through (x_2, y_2) and (x_1, y_1).

$$\text{Slope of } P_1\,P_2 = \frac{B}{A}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{B}{A}$$

Or $$(y_2 - y_1) = \frac{B}{A}\,(x_2 - x_1)$$

$$Ay_2 - Ay_1 = Bx_2 - Bx_1$$

Or $$Bx_2 - Ay_2 + Ay_1 - Bx_1 = 0 \qquad (3)$$

Now solving equation (2) and (3) for x_2 and y_2

$$Ax_2 + By_2 + C = 0 \qquad (2)$$

$$Bx_2 - Ay_2 + Ay_1 - Bx_1 = 0 \qquad (3)$$

$$\frac{x_2}{ABy_1 - B^2x_1 + AC} = -\frac{y^2}{A^2y_1 - ABx_1 - BC} = \frac{1}{-A^2 - B^2}$$

$$x_2 = \frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$$

$$y_2 = \frac{A^2y_1 - ABx_1 - BC}{A^2 + B^2}$$

Putting x_2 and y_2 in equation (1)

$$|P_1\,P_2| = \sqrt{\left(\frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2} - x_1\right)^2 + \left(\frac{A^2y_1 - ABx_1 - BC}{A^2 + B^2} - y_1\right)^2}$$

$$= \sqrt{\left(\frac{-ABy_1 - AC - A^2x_1}{A^2 + B^2}\right)^2 + \left(\frac{-ABx_1 - BC - B^2y_1}{B^2 - A^2y_1 - B^2y_1}\right)^2}$$

$$= \sqrt{\frac{\{-A\,(Ax_1 + By_1 + C)\}^2\ \} + \{\ -B\,(Ax_1 + By_1 + C)\}^2}{(A^2 + B^2)^2}}$$

$$= \sqrt{\frac{A^2\,(Ax_1 + By_1 + C)\}^2 + B^2\,(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}}$$

$$= \sqrt{\frac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}}$$

Or $d = |P_2\,P_2| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ (4)

Which is the required distance from a point $P_1\,(x_1, y_1)$ on the line $Ax + By + C = 0$

Remarks

If we take the expression

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$$

The Numerator of this expression i.e., $Ax_1 + By_1 + C$ will be positive, negative or zero depending upon the relative positions of the point P_1, the line, and the origin. If $P_1\,(x_1, y_1)$ is any point, and

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$$

has the same sign as $\dfrac{C}{\sqrt{A^2 + B^2}}$

Then $P_1\,(x_1, y_1)$ and the origin are on the same side of the line. If the signs are different, then $P_1\,(x_1, y_1)$ is on the opposite side of the line from the origin.

For $P_1\,(0, 0)$, we get the directed distance from the origin to the line, which may be positive, negative or zero.

Example 26: Find the distance from the point (–3, 2) to the line 2x – y + 4 = 0, Is (–3, 2) on the same side of the line as the origin, or is it on the opposite side?

Solution:

$$d = \frac{|2(-3) - 2 + 4|}{\sqrt{2^2 + (-1)^2}} = \frac{|-4|}{\sqrt{5}}$$

$$d = \frac{4}{\sqrt{5}}$$

Because $\dfrac{Ax_1 - By_1 + C}{\sqrt{A^2 + B^2}} = \dfrac{-4}{\sqrt{5}}$

and $\dfrac{C}{\sqrt{A^2 + B^2}} = \dfrac{4}{\sqrt{5}}$

has different signs, so the point (–3, 2) and the origin are on opposite side of the line $2x - y + 4 = 0$ as shown in Fig. 20.

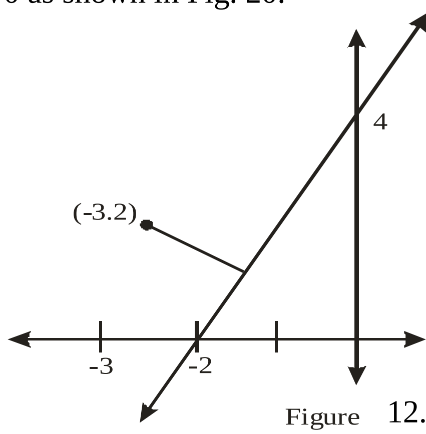


Figure 12.20

Example 27: An equation of a line is $4x - 3y + 12 = 0$. Find the coordinates of the point P_o (x_o, y_o) which is the foot of the perpendicular from the origin to the line.

Solution:

Since point P_o (x_o, y_o)

Lies on the line $4x - 3y + 12 = 0$

So,

$4x_o - 3y_o + 12 = 0$ …………..(1)

The slope of line (1) is

$y_o = \frac{4}{3}\ x_o + 4$

$m\ = \frac{4}{3}$

The slope of the perpendicular $\overline{OP}$ is

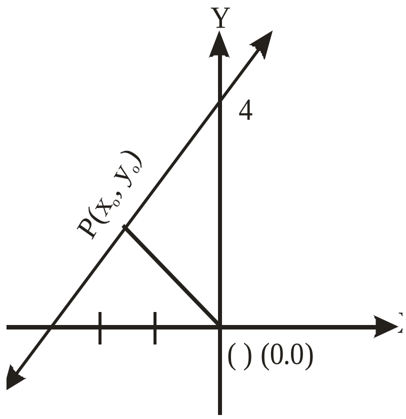


Figure 12.21

$$\frac{y_o - 0}{x_o - 0}\ = \ -\frac{3}{4}$$

Or $\quad 3x_o + 4y_o = 0$ ……………………..(2)

Solving equation (1) and (2) together, we find

$$\frac{x_o}{0 - 48} = \ -\frac{y_o}{0 - 36} = \frac{1}{16 + 9}$$

$$x_o \quad = -\frac{48}{25}\ , \qquad y_o = \frac{36}{25}$$

Hence the coordinates of P_o are : $\left(-\frac{48}{25} \quad \frac{36}{25}\right)$

Example 28: If P is the perpendicular distance of the origin from a line whose intercepts on the axes are a and b, show that.

$$\frac{1}{P^2}\ = \frac{1}{a^2}\ + \ \frac{1}{b^2}$$

Solution:

The equation of the line in intercepts form is

$$\frac{x}{a} = \frac{y}{b} = 1$$

Or $\quad xb + ay - ab = 0$

If P is the perpendicular distance from the origin 0 (o,o) on the line bx + ay – ab = 0.

Then $\quad P = \frac{|b(o) + a(o) - ab|}{\sqrt{b^2 + a^2}} = \frac{|-ab|}{\sqrt{b^2 + a^2}}$

Or $\quad P = \frac{ab}{\sqrt{b^2 + a^2}}$

Or $\quad P^2 = \frac{a^2\, b^2}{b^2 + a^2}$

Or $\quad \frac{1}{P^2} = \frac{a^2 + b^2}{b^2\, a^2}$

Or $\quad \frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Example 29: Find the equations of the two lines (or find the locus of a point) which are parallel to and 3 units from the perpendicular bisector of the line segment (1, – 2), (– 3, 8).

Solution:

Mid-point of the segment (1, **–3)**

Slope of the segment is:

$$m = \frac{8 + 2}{-3 \ -1} \quad = \quad \frac{10}{-4}$$

$$m = -\frac{5}{2}$$

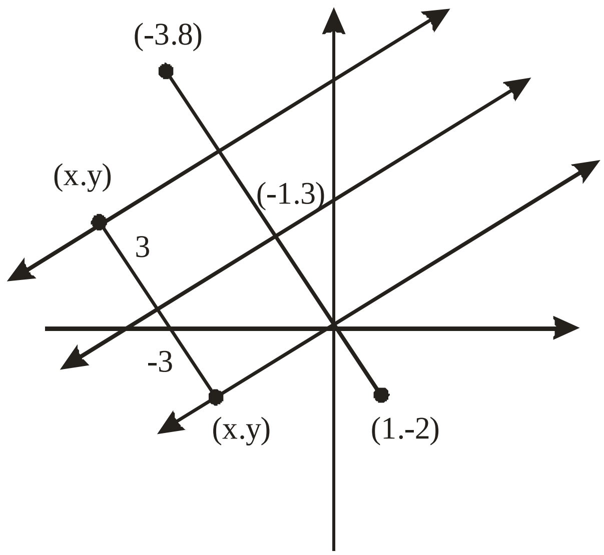


Figure 12.22

Slope of the perpendicular bisector is $m_1 = \frac{2}{5}$

Equation of the perpendicular bisector passing through (–1, 3)

$$y - 3 = \frac{2}{5}\ (x + 1)$$

$$5y - 15 = 2x + 2$$

$$2x - 5y + 17$$

Since the required equations are of a distance of 3 units from the perpendicular bisector $2x - 5y + 17 = 0$. So by the formula.

$$d = \frac{|Ax_1 + By + C|}{\sqrt{A^2 + B^2}}$$

$$3 = \frac{|2x - 5y + 17|}{\sqrt{2^2 + (-5)^2}}$$

$$3 = \frac{|2x - 5y + 17|}{\sqrt{29}}$$

Or $|2x - 5y + 17| = 3\sqrt{29}$

Or $2x - 5y + 17 = \pm 3\sqrt{29}$

So the equation (for locus) are

$$2x - 5y + 17 - 3\sqrt{29} = 0$$

and $2x - 5y + (17 + 3\sqrt{29}) = 0$

Exercise 12.6

Q.1: Find the distance to the line $3x - 2y + 12 = 0$ from each of the following points:
(a) (1, 3) (b) (–1, 7) (c) (–3, –2)

Q.2: Which of the following points are on the same side of the line $x - 6y + 8 = 0$ as the origin?
(a) (2, 3) (b) (3, –2) (c) (–2, –3) (d) (–3,2)

Q.3: If the vertices of a triangle are A (2, –1), B(–2,3) and C(0,3), Find the length of altitude from B to AC.

Q.4: The distance from the point (–2, –3) to the line $3x - 4y + k = 0$ is $\frac{13}{5}$. Find the value of K.

Q.5: Write equation of the two lines (or locus of a point) parallel to the line through (1, 2) and (4, 6) which are 3 units distant from the given line.

Q.6: Find the locus of a point which moves so that it is always "one" unit from the line $3x - 4y + 7 = 0$.

Q.7: Find the equation of the two lines (or locus of point) parallel to the line $x - 6y + 8 = 0$ and a distance of $\frac{18}{\sqrt{37}}$ units from it.

Q.8: Find the distance between the parallel lines.

$3x - 4y + 11 = 0$ and $3x - 4y - 9 = 0$

(Hints: Take a point at one line and find the distance of this point on the other line)

Q.9: Find the perpendicular distance from the origin to the line passing through (1, 2) and perpendicular to the line $\sqrt{3}\ y = x + 4$.

Q.10: Find the locus of all points which are equidistant from the point (– 3, 8) and the line $4x + 9 = 0$

Answers

Q.1: (a) (a) $\dfrac{9}{\sqrt{13}}$ (b) $\dfrac{5}{\sqrt{13}}$ (c) $\dfrac{12}{\sqrt{13}}$ (d) $\dfrac{7}{\sqrt{13}}$

Q.2: (b), (c) **Q.3**: $4\sqrt{2}$ **Q.4:** $k = 7, -19$

Q.5: $4x - 3y - 13 = 0$, $4x - 3y + 17 = 0$

Q.6: $3x - 4y + 2 = 0$, $3x - 4y + 12 = 0$

Q.7: $x - 6y - 10 = 0$, $x - 6y + 26 = 0$

Q.8: 4 **Q.9:** $\dfrac{2 + \sqrt{3}}{2}$

Q.10: $16y^2 - 256y + 24x + 1087 = 0$

Short Questions

Write the short answers of the following:

Q.1: Write distance formula between two points and give one example.

Q.2: Find distance between the points (-3, 1) and (3, -2)

Q.3: Show that the points A(-1, -1), B (4, 1) and C(12, 4) lies on a straight line.

Q.4: Find the co-ordinate of the mid point of the segment P_1 (3,7), P_2 (-2, 3).

Q.5: Find the co-ordinates of the point P(x,y) which divide internally the segment through P_1 (-2,5) and P_2 (4, -1) of the ratio of $\frac{r_1}{r_2} = \frac{6}{5}$.

Q.6: If a line is extended from A(2, 3) through B(-2, 0) to a point C so that AC = 4 AB, find the co-ordinate of C.

Q.7: For the triangle whose vertices are A(0,1), B (7,2) and C(3,8). Find the length of the median from C to AB.

Q.8: If the mid point of a segment is (6,3) and one end point is(8, -4), what are the co-ordinates of the other end point.

Q.9: Find the angle between the lines having slopes -3 and 2

Q.10: Find the slope of a line which is perpendicular to the line joining P_1 (2, 4) and P_2 (-2, 1).

Q.11: Find the equation of a line through the point (3, -2) with slope $m = \frac{3}{4}$.

Q.12: Find the equation of a line through the points (− 1, 2) and (3, 4).

Q.13: Find an equation of the line with the following intercepts $a = 2$, $b = -5$

Q.14: Find the equation of line having x – intercept -2 and y – intercept 3.

Q.15: Find the equation of a line whose perpendicular distance from the origin is 2 and inclination of the perpendicular is 225^o.

Q.16: Reduce the equation $3x + 4y - 2 = 0$ into intercept form.

Q.17: Find the equation of the line passing the point (1, –2) making an angle of 135^o with the x-axis.

Q.18: Find the points of intersection of the lines
$x + 2y - 3 = 0, 2x - 3y + 8 = 0$

Q.19: Show that the points (1,9), (-2, 3) and (-5, -3) are collinear.

Q.20: Show that the lines passing through the points (0, -7), (8, -5) and (5, 7), (8, -5) are perpendicular.

Q.21: Find the distance from the point (-2,1) to the line $3x + 4y - 12 = 0$

Answers

Q2.	$3\sqrt{5}$	**Q4.**	$\left(\frac{1}{2}, 5\right)$	**Q5.**	$P\left(\frac{14}{11}, \frac{19}{11}\right)$
Q6.	C (– 14, – 9)	**Q7.**	$\sqrt{\left(\frac{85}{2}\right)}$	**Q8.**	(4, 10)
Q9.	135^o	**Q10.**	$-\frac{4}{3}$	**Q11.**	$3x - 4y - 17 = 0$
Q12.	$x- 2y + 5 = 0$	**Q13.**	$5x - 2y = 10$	**Q14.**	$3x - 2y + 6 = 0$

Q15. $x + y + 2\sqrt{2} = 0$

Q16. $\frac{x}{2/3} + \frac{y}{1/2} = 1$, where $a = \frac{2}{3}$ and $b = \frac{1}{2}$

Q17. $x + y + 1 = 0$ Q18. (– 1, 2) Q21. $\frac{14}{5}$

<><><>

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Slope of the line $\frac{x}{a} + \frac{y}{b} = 1$ is:

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) $-\frac{b}{a}$ (d) $-\frac{a}{b}$

__2. y = 2 is a line parallel to:

(a) x – axis (b) y – axis (c) y = x (d) x = 3

__3. Eq. of the line in slope intercept form is;

(a) $\frac{x}{y} + \frac{y}{b} = 1$ (b) $y = mx + c$ (c) $y - y_1 = m(x-1)$

(d) None of these

__4. Distance between (4, 3) and (7, 5) is:

(a) 25 (b) $\sqrt{13}$ (c) 5 (d) None of these

__5. Point (– 4, –5) lines in the quadrant:

(a) 1^{st} (b) 2^{nd} (c) 3^{rd} (d) 4^{th}

__6. When two lines are perpendicular:

(a) $m_1 = m_2$ (b) $m_1m_2 = -1$ (c) $m_1 = -m_2$

(d) None of these

__7. Ratio formula for y – coordinate is:

(a) $\frac{x_1r_2 + x_2 r_1}{r_1 + r_2}$ (b) $\frac{y_1r_2 + y_2 r_1}{r_1 + r_2}$ (c) $\frac{x - y}{2}$

(d) None of these

__8. Slope of the line through (x_1 , y_2) and (x_2, y_2)

(a) $\frac{x_1 + x_2}{y_1 + y_2}$ (b) (c) $\frac{y_2 - y_1}{x_2 - x_1}$

(d) None of these

__9. Given three points are collinear if their slopes are:

(a) Equal (b) Unequal (c) $m_1 m_2 = -1$

(d) None of these

__10. $y - y_1 = m(x - x_1)$ is the:

(a) Slope intercept form (b) Intercept form

(c) Point slope form (d) None of these

Answers

1.	c	**2.**	a	**3.**	b	**4.**	b	**5.**	c
6.	b	**7.**	b	**8.**	c	**9.**	a	**10.**	c